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PROCEEDINGS
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MAGELLANIC CLOUDS. I. TRANSPARENCY

By Harlow Shapley

Harvard College Observatory

Communicated January 10, 1951

Several special studies of the Clouds of Magellan have been undertaken in the past year as a part of a monographic treatment of these nearest of external galaxies. One such investigation is the examination of the extent to which they are transparent to the light of more distant stellar systems. They are sufficiently large in angular extent that a deficiency in the population of faint background galaxies is a satisfactory indication, and a rough measure, of the internal space absorption. The amount of gas and dust now in the Magellanic Clouds is obviously of importance to speculations on their origin and development. A few of the larger spirals also should similarly reveal their total absorption, at least in their uncrowded outer regions, when searched for background galaxies to the twenty-first magnitude and fainter.

For the Bruce refractor, with which the Harvard survey of southern galaxies has been made, the scale is too small to differentiate clearly in many instances the distant translucent galaxies in the fields of the Clouds from their own numerous nebulosities, star clusters, and nebulous groups. But the scale of the 60-inch reflector, 1 mm. = 26.″6, suffices to clarify many of the doubtful objects.

The whole of both Magellanic Clouds and some of the surrounding areas have been covered with reflector plates, mostly of thirty minutes' exposure on fast emulsions. Only the central part with undistorted images—a fifth of a square degree—has been searched on each plate for external galaxies. Because of the large zenith distances of the Clouds, even when on the meridian ($z = 44°$, and $z = 40°$ for Small and Large Cloud, respectively), many of the plates are of low quality. For the Small Cloud, therefore, only 60 of the better plates, covering a total of 12 square degrees, have been used in this examination; for the Large Cloud, 122 plates are used, covering 24.4 square degrees. Reflector plates of comparable exposure in regions

well outside the Clouds provide something of a check on the surety of identifying faint galaxies and on their frequency, but the total area outside the Clouds covered with the reflector plates is too small to justify a direct quantitative comparison with the populations in the Clouds. For the quantitative comparison of the Clouds with the neighboring fields, the large-area Bruce plates are used. They also serve for the estimates of total magnitudes of the galaxies.

The comparison fields around the Clouds are selected from a published catalog of galaxies from declination $-60°$ to the South Pole.[1] They are chosen in such a manner (table 1) that the mean galactic latitudes and longitudes fall within the Clouds they surround. Each plate covers nine square degrees. From the last column of the table, which records the number of galaxies per square degree brighter than magnitude 17.5, it is found that the surroundings of the Small Cloud are considerably below normal population ($N = 21$), and around the Large Cloud, somewhat richer than the average high latitude field. Also, for both the Clouds, the surrounding fields are of uneven population; in consequence this test of the transparency does not have high quantitative value.

TABLE 1

COMPARISON FIELDS

	FIELD	PLATE NUMBER	GALACTIC LONG. °	GALACTIC LAT. °	NUMBER OF GALAXIES
Small cloud	1	A14232	273	−50	8.7
	2	17198	268	−40	6.1
	3	17222	264	−50	18.6
	4	17150	257	−45	12.3
	5	14395	261	−41	7.8
	6	14213	284	−48	7.2
	7	16299	280	−44	4.4
	8	17194	273	−40	8.1
	Means		270	−45	9.1
Large cloud	1	17155	250	−44	9.8
	2	17163	253	−38	48.3
	3	17187	243	−39	59.0
	4	16686	254	−34	27.2
	5	16732	239	−37	60.9
	6	16641	239	−30	15.4
	7	16368	250	−28	15.2
	8	16758	245	−26	18.6
	9	14256	240	−25	18.1
	Means		246	−33	30.3

The excess of galaxies in the neighborhood of the Large Cloud has long been known. The Cloud lies, in fact, near the end of an extensive meta-

galactic cloud of faint galaxies, that extends from the Large Cloud some forty degrees or more in the direction of the South Galactic Pole.[2]

The poorness of the environment of the Small Cloud, notwithstanding its high galactic latitude of $-45°$, is somewhat surprising. The question naturally arises as to whether in this direction there is a cloud of interstellar material, perhaps near the observer, that dims the light of the galaxies by half a magnitude. That possibility, however, is ruled out by the star counts which show, in the van Rhijn tables, for instance, a stellar density around the Small Cloud that is not at all below average for the galactic latitude. A blocking dust cloud outside our Galaxy, in intergalactic space, which would dim galaxies but not stars, would need to be of unacceptably large dimensions and mass. It seems best to interpret the scarcity of galaxies in the general region of the Small Cloud to a detail of metagalactic structure. There are similar sparse regions elsewhere in both the northern and southern skies. Long exposure photographs around the Clouds with the Baker-Schmidt telescope in South Africa, which will take the nebular counts to the nineteenth magnitude or fainter, will eventually clarify the situation.

TABLE 2

GALAXIES PER SQUARE DEGREE

	NUMBER PLATES	AREA, SQ. DEG.	TO MAGNITUDE LIMIT 15.95	16.45	16.95	17.45	17.95
Small cloud							
Inner	15	3	0.3	1.3	4.3	7.7	11.3
Outer	45	9	0.2	1.0	3.9	9.3	17.3
Whole cloud	60	12	0.2	1.1	4.0	8.9	15.8
Comparison fields	8	72	0.5 ± 0.2	1.5 ± 0.2	3.6 ± 0.5	9.1 ± 1.6	16.3 ± 2.6
Large cloud							
Axis	18	3.6	0.5	1.7	3.3	3.6	3.9
Clustered areas	49	9.8	0.2	1.6	3.0	5.9	11.4
Open areas in cloud	42	8.4	0.1	0.7	2.6	4.2	10.7
Borders of cloud	13	2.6	0.0	1.2	4.6	12.7	29.2
Whole cloud	122	24.4	0.2	1.3	3.1	5.7	12.0
Comparison fields	9	81	1.0 ± 0.1	2.9 ± 0.2	10.0 ± 1.9	30.4 ± 6.7	77.6 ± 22.6
Sparse comparison fields	4	36	1.0 ± 0.2	2.4 ± 0.3	7.4 ± 1.4	14.9 ± 1.7	25.0 ± 2.1

In table 2 the results from the reflector plates are assembled. The Small Cloud is divided into an inner and outer region; the Large Cloud is examined separately for four regions—the axis; the clustered areas, which are rich in giant clusters, smaller nebulous groups, and bright and dark nebulosity; the open regions (stellar deserts) inside the main body of the Cloud; and the open regions that lie around the border of the main body but within the area where outlying Cepheids and clusters are found. The cosmograph in figure 1 indicates the location of these four regions.[3]

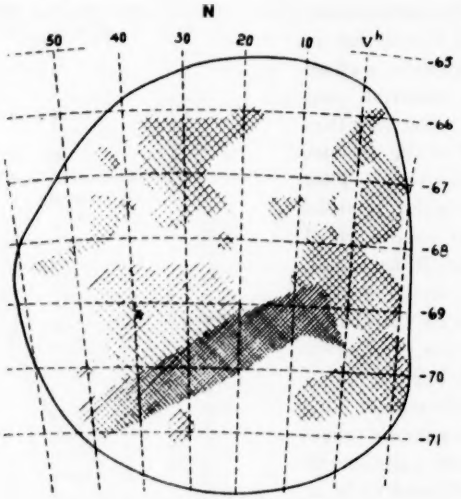

FIGURE 1

The cosmograph of the Large Magellanic Cloud shows in the main body of the system, the Axis and the coarsely cross-hatched clustered regions, as well as the more thinly populated deserts where also the translucent galaxies are dimmed by approximately a magnitude.

The numbers of plates are given in the second column of table 2—reflector plates for the Clouds, Bruce plates for the comparison fields. The areas of the various regions and the number of galaxies per square degree to successive limits of magnitude are given in the following columns. For the comparison fields the mean errors refer not to square degrees but to the individual plates, each of nine square degrees area.

Conclusions.—An examination of the tabulated numbers and of figure 2 shows that the Small Magellanic Cloud is essentially transparent. For the translucent galaxies brighter than magnitude 17.25 both the inner and outer

regions show a population closely comparable with that in the eight surrounding large areas, but the numbers of objects involved are small. For the magnitude interval 17 to 18, where there would naturally be some difficulty in finding translucent galaxies in the core of the Cloud because of the relatively high star density, the inner population falls below that of the outer region of the Cloud, which in turn seems to be a little more populous than the mean of the comparison fields. An interstellar absorption in the inner section of the Cloud of two- or three-tenths of a magnitude is not out of the question. There are several small nebulous wisps in the central part of the Cloud and probably some scattering dust associated with these bits of nebulosity.

The observations in the Large Cloud region, in table 2 and figure 3,

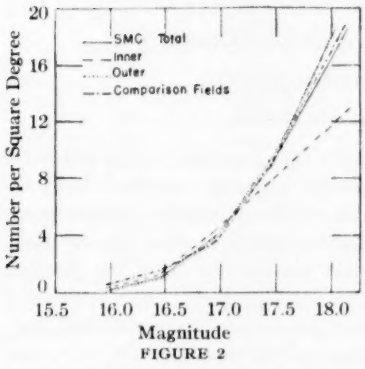

FIGURE 2

Distribution of galaxies in and around the field of the Small Magellanic Cloud.

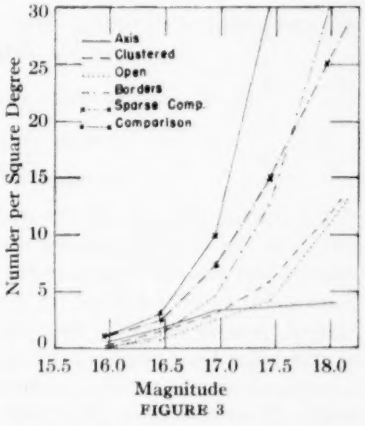

FIGURE 3

Distribution of galaxies in and around the field of the Large Magellanic Cloud.

give a conspicuously different result. The crowded axis is nearly devoid of translucent galaxies. Galaxies are also scarce in clustered areas, and, surprisingly, in the open areas in the middle of the Cloud. On the borders of the main body of the Cloud, the frequency is about the same as on the four sparsely populated comparison field plates, which show a population much like the average for high latitudes the whole sky over.

From these data we conclude that space absorption apparently can be ignored on the borders of the Large Cloud, but amounts to as much as a magnitude at some places inside. The presence of much visible nebulosity and also of very many supergiant stars is consistent with this result on opacity, and suggests that the birth of low-density high luminosity stars may be a current operation in the Large Cloud, whereas in the essentially

dust-free Small Cloud the age of star birth is past. This hypothesis rests, of course, on the assumption that under appropriate conditions the interstellar medium can aggregate into new stars of vast extent.

The two Clouds also differ in certain properties of their Cepheid variables. In a forthcoming paper this difference will be related to the transparency peculiarities.

[1] *Harv. Ann.*, **105**, No. 8, 140–141 (1937).

[2] These PROCEEDINGS, **24**, 282–287 (1938); *Harv. Rep.* 150 (1938); *Ibid.*, **24**, 527–531 (1938); *Harv. Rep.* 154 (1938).

[3] *Harv. Obs. Bull.* 881, 1 (1931).

MAGELLANIC CLOUDS. II. SUPERGIANT RED VARIABLE STARS IN THE SMALL CLOUD

BY HARLOW SHAPLEY AND VIRGINIA MCKIBBEN NAIL

HARVARD COLLEGE OBSERVATORY

Communicated January 17, 1951

How bright and how big can stars be? This is a question for which the nearer of the external galaxies can provide a precise answer. Both the relative and absolute values of the total outflow of energy of super-luminous stars of various types, in any discrete stellar system, can be found simply from measures of their apparent magnitudes, once the distance modulus has been determined from Cepheid variables. (The interstellar dimming in our own Galaxy and between galaxies does not enter the problem.) From the luminosities and colors we can also compute the corresponding gigantic sizes of these brightest of stars in external galaxies.

Of the nearer galaxies, the Large and Small Clouds of Magellan are of course the most useful for an inquiry into the frequency and behavior of supergiants. These galaxies are rich in various types of stars and structurally are wide open to photometric and spectroscopic surveys, even with telescopes of intermediate size. The high luminosity stars in our own galaxy, on the other hand, cannot be comprehensively surveyed. With us the uncertainties in distances, the uneven dimming by space absorption, and the incomplete census in remote regions prevent the making of a good survey.

1. Supergiant Luminosities.—In the present communication we report on high-luminosity reddish variable stars of the Small Cloud, listing all the known long-period variables and irregular or semiregular stars brighter photographically than absolute magnitude -2.3 at maximum.* A later study of these types of variables will be based on spectroscopic studies.

2.—The brightest star in the universe so far as now recorded is the well-known bluish variable star of the Large Magellanic Cloud, S Doradus. Its absolute magnitude† appears to be about −9 (half a million times the Sun's luminosity), and its brightness is therefore exceeded only by the bursts of super-novae. Doubtless there are similar superlatively bright stars in the open spirals and in the more massive of the irregular galaxies. In the Small Cloud no variable is so bright. Among the Cloud's constant stars there are, in our records to date, seven of spectral class B with absolute magnitudes ranging from −6.8 to −5.7, two Class O stars with absolute magnitudes of about −5, and one P-Cygni star with absolute magnitude −6. (For our galactic system the absolute magnitudes of the irregular

TABLE 1

DESIGNATION	TYPE	PERIOD IN DAYS	ABS. MAG. AT MAX.	RADIUS IN A.U.
HV 1956	Cepheid	209	−5.5	8.3
821	Cepheid	127	−5.0	6.6
824	Cepheid	65.8	−5.0	6.6
834	Cepheid	73.5	−5.0	6.6
2195	Cepheid	41.8	−4.7	5.8
829	Cepheid	88.5	−4.6	5.5
11423	Irregular	...	−4.5	5.3
1475	Irregular	...	−4.2	4.6
11157	Cepheid	64.8	−4.1	4.4
863	Cepheid	29.0	−4.1	4.4
837	Cepheid	42.6	−4.1	4.4
847	Cepheid	27.2	−4.1	4.4
865	Cepheid	33.3	−4.1	4.4
840	Cepheid	33.0	−4.0	4.2
817	Cepheid	18.9	−3.9	4.0
1541	Cepheid	19.3	−3.9	4.0
1652	Irregular	...	−3.8	3.8
1636	Cepheid	32.7	−3.8	3.8
2084	Irregular	...	−3.7	3.6
2064	Cepheid	33.7	−3.7	3.6

red variables have been discussed by Joy,[1] and by Wilson,[2] and of the long-period variables by Wilson and Merrill.[3] The highest luminosities at maximum do not exceed photographic magnitude −3.0 and very few exceed −2.0.)

3.—In the recognized membership of the Small Cloud the twenty brightest variables are those of table 1. HV 1956, at the top of the list, may of course be a foreground star, but it conforms precisely with the period-luminosity relation.

4. Supergiant Sizes.—Not only the greatest radiators but also stars of largest known dimensions are revealed by the survey of reddish supergiants in external galaxies. The long-period variables, the supergiant

TABLE 2

CENSUS IN VARIABLE STAR FIELDS WITH $\beta > +30°$

FIELD*	GALACTIC LAT.	GALACTIC LONG.	BR. THAN 12 LP	BR. THAN 12 IRR	12 TO 13 LP	12 TO 13 IRR	13 TO 14 LP	13 TO 14 IRR	14 TO LIMIT LP	14 TO LIMIT IRR
209	−40	0	2	4	0	4	0	6	0	0
211	−40	170	1	1	1	2	1	3	0	1
212	−40	270	0	3	1	2	2	3	0	2
10	+40	270	4	2	1	1	0	0	0	0
524	−41	255	4	4	1	4	1	6	0	3
566	−46	245								
		SUBTOTAL	11	14	4	13	4	18	0	6
Expected from Foreground T/16			0.7	0.9	0.3	0.8	0.3	1.0	0.0	0.4
Found in Small Cloud			0	1	0	1	0	4	11	25
356	+31	270	2	2	4	0	4	4	1	3
213	−60	0	2	0	1	1	1	0	0	0
214	−60	95	1	0	0	1	0	2	0	0
215	−60	180	1	0	0	0	0	0	0	0
216	−60	270	2	0	0	0	0	3	0	0
6	+62	270	3	1	0	1	0	0	0	0
217	−80	180	1	0	0	0	0	0	0	0
218	−80	356	1	0	1	1	1	1	0	0
		TOTAL	24	17	10	17	10	28	1	9
Expected from Foreground T/40			0.6	0.4	0.2	0.4	0.2	0.7	0.0	0.2
Found in Small Cloud			0	1	0	1	0	4	11	25

* For summary of references see Walter J. Miller, S.J., "The Harvard Variables Star Fields," *Ricerche Astronomiche*, Vol. 1, no. 11 (1946); Harvard Reprint 290 (1947).

Cepheids, and the reddish irregular and semiregular variables of the Small Cloud all have low radiative efficiency, and therefore those of highest absolute luminosity have surface areas far in excess of the largest stars of similar types yet measured in our galactic system. To illustrate the sizes, let us take c.i. = +1.5 magnitudes as a reasonable average value for the color index of the irregular red variables of the Small Cloud, which resemble Betelgeuse in variability and in color, and presumably in surface brightness. Then the *visual* absolute magnitudes of the five brightest in table 5 (omitting HV 11546, which may be superposed) average $M_v = -4.0 - \text{c.i.} = -5.5$ at maximum. They are therefore $\Delta M = 2.0$ magnitudes brighter absolutely than the mean of Betelgeuse and Antares,[4] and their radii must be approximately $10^{0.2\,\Delta M} = 2.5$ times as great. Such stars would fill the orbit of Jupiter. In the Large Cloud we shall find larger and brighter red stars. Probably such supergiants, though yet undiscovered, exist also in our own galactic system.

Comparing directly with Betelgeuse, of absolute magnitude −3.9 (vis.) and a radius r_B of two astronomical units, and assuming a color index of +1.5, we compute the radii $r_v = r_B 10^{0.2\,\Delta m}$ of the twenty stars in table 1 and obtain the values, in astronomical units, given in the last column. The mean is 4.9; the radius of Jupiter's orbit, 5.2 astronomical units, is exceeded by the radii of seven of the variables.

5. The Comparison Fields.—The large angular dimensions of the Magellanic Clouds make necessary a special investigation before actual membership rather than random superposition can be assigned to the variable stars. For some types of stars, however, the high galactic latitudes of the Clouds assist in the discrimination between Cloud members and superposed stars of the galactic system. The high concentration toward the galactic plane of classical Cepheids, novae, open star clusters, bright nebulosities, and stars of spectral classes B, O, and P-Cygni, gives assurance that when such objects appear on photographs of the Clouds they are actual Cloud members. On the other hand, for the eclipsing binaries of most types, long-period and irregular variable stars, all of which have a wide distribution in galactic latitude, we need to make a closer study. Two methods of discriminating are available; the radial velocities, and the intercomparison of populations in and out of the Clouds. The radial velocities of the Clouds are high and are rarely equalled by galactic variables; but the apparent faintness of the stars in the Clouds has up to now excluded the use of the spectrographic test on the variable stars. We use the other method.

Fourteen comparison fields, each of about 75 square degrees, have been thoroughly examined for variable stars. Table 2 lists the fields and for each the number of variable stars of the long-period and irregular types, in four intervals of magnitude. For the work on the Small Cloud ($\beta = -45°$) we have used only the six fields of similar latitudes at the top of the

list, two of which, between the Clouds, somewhat overlap. About 400 square degrees are covered by the six fields, and 1000 square degrees are covered by all fourteen. The Small Cloud variables fall within an area of 25 square degrees. Hence the subtotals and the totals of table 2 are divided by $400/25 = 16$ and $1000/25 = 40$, respectively, to give the population of foreground variables, that would be expected on the basis of the comparison field surveys in the area covered by the Small Cloud.

In itself, this census of high latitude fields has some points worth noting. For example, long-period variables do not appear fainter than the fourteenth magnitude, with the exception of a single variable in the field of lowest latitude (No. 356). In fact, except in that field, there are only six long-period variables fainter than magnitude 13 in more than 900 square degrees. In the seven polar cap fields (latitudes 60° or higher, area 525

TABLE 3

LONG-PERIOD VARIABLES: PROBABLE CLOUD MEMBERS

HARVARD VARIABLE	MAXIMUM MAGNITUDE	MAGNITUDE RANGE	ABSOLUTE MAGNITUDE	PERIOD IN DAYS	DISTANCE FROM CENTER	COLOR
859	14.3	]2.7	−3.0	573.4	1°2	Red
1719	14.5	]2.4	−2.8	531.9	0.5	Red?
2112	14.2	]2.8	−3.1	607.2	1.2	Very red
12149	14.5	3.6	−2.8	740.7	0.4	Red?
1375	15.8	]1.3	−1.5	510	1.5	Red
1645	16.3	]0.8	−1.0	300.3	0.7	Yellow?
1865	15.7	]2.3	−1.6	562.4	0.6	Red
11295	15.0	1.8	−2.3	571.4	0.4	Yellow
11303	15.7	]1.3	−1.6	540.0	1.0	Red
11329	15.2	1.9	−2.1	388.0	0.3	Very red
11427	16.0	]1.0	−1.3	254.4	0.6	Red

square degrees) there are only four long-period variables fainter than the twelfth magnitude at maximum. In other words, the long-period variables, of which hundreds are known, are rarely found more than a thousand parsecs from the galactic plane. Irregular variables are found more numerously among the fainter stars.

From table 2 we conclude that any long-period variable fainter than 12.0 at maximum, appearing on Cloud photographs, is almost certainly a member of the Cloud. Brighter than the twelfth magnitude we might expect one superposed long-period variable and one irregular variable. Possibly in the magnitude interval 12–14 one or two superposed irregular variables might be expected but, fainter than 14, probably nothing from the foreground.

In summary, we find from the study of fourteen comparison fields in similar galactic latitudes that practically every reddish variable star of

these two important types fainter than 11.0, discovered on the photographs of the Clouds, are actual Cloud members and therefore giants and supergiants.

6. Long-Period Variables.—In 1923 the three brightest variable stars in the southern globular cluster 47 Tucanae were shown to be long-period variables of extraordinarily high luminosity at maximum. Their crowding toward the center of the cluster argued against the possibility of their being superposed galactic stars. The long-period variable stars later detected in the adjacent Small Cloud could not be assigned so definitely to the supergiant category until the completion of the comparison field work described in the preceding section. The assignment is now possible. Eleven of the bright long-period variables of the Small Cloud region are actually members of the Cloud.[5] The brightest four, $\overline{M} = -2.9$, are about four magnitudes brighter absolutely than the average long-period variable in the galactic system, but little if any more luminous than the brightest of the galactic long-period variables.[3] Table 3 contains information on all the long-period variables now known in the Small Cloud, with the absolute magnitudes

TABLE 4

FOREGROUND LONG-PERIOD VARIABLES

NAME	MAXIMUM MAGNITUDE	RANGE	PERIOD IN DAYS	SPECTRUM
HV 860	10.0	6.0	215.5	*M3e*
TZ Tuc.	9.8	]3.7	228.8	*M0e*
U Tuc.	8.5	6.7	262.3	*M7e*
HV 864	11.0	4.5	319	*M5e*

based on the assumption of Cloud membership. The greatest distance from the geometrical center of the Cloud is 1°.2. Therefore the eleven variables are crowded into less than 5 square degrees.

7.—It should be noted that eight of the periods are unusually long, and for at least two of the variables the ranges are small. Although uncommon in range and period similar variables are found in the galactic system.[6]

In the vicinity of the Small Cloud are four long-period variables of typical range, period, and spectrum; all are brighter than the eleventh magnitude at maximum (table 4). Presumably they are members of the galactic system, at a distance of only a thousand parsecs or so, and therefore too near and too bright for us to estimate accurately either distance or absolute magnitude!

8. Irregular and Semiregular Variables.—In the first part of table 5 are listed the irregular and semiregular variable stars of the Small Cloud for which the apparent magnitudes at maximum are 14.9 or brighter. In the lower section are listed the other ten fainter variables of these kinds that have been sufficiently examined to determine their characteristics. As for the long-period variables, the distance from the center and the color are

given along with data on magnitudes. Four of the stars, marked with asterisks, had not heretofore been published. Their positions, in the usual coordinate system,[7] are as follows:

DESIGNATION	X	Y
b	11235″	8426″
c	11330	8860
k	12002	8212
w	12192	8310

TABLE 5

IRREGULAR AND SEMIREGULAR VARIABLES

HARVARD VARIABLE	MAXIMUM MAGNITUDE	RANGE	ABSOLUTE MAGNITUDE	NOTES	DISTANCE FROM CENTER	COLOR
825	14.4	0.7	−2.9	(1)	0.°5	Very red
832	14.2	0.6	−3.1	(1)	0.3	Very red
838	14.2	]3.0	−3.1	575^d*	0.6	Red
1455	14.5	0.7	−2.8	(1)	1.4	Very red
1475	13.1	0.7	−4.2	(2)	0.8	Very red
1652	13.5	1.2	−3.8	...	0.0	Very red
2084	13.6	1.4	−3.7	(1)	1.3	Very red
2228	14.4	1.0	−2.9	...	2.2	Red
2232	14.2	0.9	−3.1	(1)	2.7	Very red
11262	14.7	0.7	−2.6	Slow	0.7	Very red
11280	14.9	0.8	−2.4	...	0.5	Red
11373	14.8	0.5	−2.5	Slow	0.4	Yellow
11401	14.5	1.2	−2.8	...	0.5	Yellow
11402	13.7	1.1	−3.6	...	0.5	Very red
11423	12.8	1.0	−4.5	...	1.0	Red
11465	14.5	0.7	−2.8	Slow	0.7	Very red
11470	14.9	0.6	−2.4	Slow	0.7	Yellow
11546	11.3	0.8	−6.0	...	2.6	Red
b*	14.4	0.6	−2.9	Slow	0.7	Blue
c*	14.1	0.8	−3.2	Slow	0.6	Red
k*	14.2	0.6	−3.1	Slow	0.6	Red
1432	15.8	1.2	−1.5	...	1.0	Very red
1586	15.7	0.9	−1.6	...	0.1	Red
1644	15.0	1.3	−2.3	200± (3)	0.8	Red?
1722	15.9	1.1	−1.4	...	0.6	Very red
1963	15.7	1.3	−1.6	...	0.7	Red
2105	15.8	0.6	−1.5	Slow	1.2	Very red
11417	15.5	2.5	−1.8	1000± (3)	0.5	Very red
11455	15.4	0.8	−1.9	...	0.8	Very red
12122	15.0	]2.5	−2.3	...	0.5	Red
w*	15.3	2.2:	−2.0	750± (3)	0.6	

* Found by Miss Henrietta H. Swope on Columbia-Vale plates. Type and color by Miss Swope; magnitudes and type verified by Mrs. Nail with Bruce plates.

(1) Hoffleit, D., *Harv. Obs. Bull.*, 900 (1935).

(2) Period of $619^d.5$, *Harv. Circ.*, 280 (1925) could not be verified.

(3) Heights of maxima differ from cycle to cycle.

As indicated in the fifth column of table 5 under "Notes," a number of these irregular variables exhibit slow variation, and for four of them an approximate length of the cyclic variation is found. The eleventh magnitude variable HV 11546 exceeds all others by more than a magnitude at maximum. It may be a foreground star. It was omitted in the listing of brightest luminosities in section 3.

9. *Summary.*—Luminosities as high as $M_{pg} = -5.0$ and diameters in excess of eight astronomical units are found in the survey of the supergiant red variable stars in the Small Magellanic Cloud. All the brightest irregular and long-period variables (21 and 4, respectively, brighter than $m_{pg} = 15.0$) are discussed in this paper; the long period Cepheids have been treated in various earlier reports.[8] Through the study of the variable star population in comparable latitudes an estimate of the number of superposed foreground variables of all types is possible. Essentially all red variables are certainly Cloud members, as are the classical Cepheids. The census of the fourteen comparison star fields shows that typical long-period variables will rarely if ever be found in latitudes higher than $\pm 40°$ if fainter than magnitude 14.0.

* The modulus is taken as 17.3 in the following discussion, and the magnitude scale is that set up with the Bruce refractor. The forthcoming revision of scale and zero-point through Mr. Ivan King's photoelectric measures will somewhat modify these values but not affect appreciably the arguments based on them.

† All magnitudes referred to in these papers on the Magellanic Clouds are photographic unless otherwise indicated.

[1] Joy, A. H., *Mt. Wilson Contr.*, 668 (1942); *Ap. J.*, **96**, 344–370 (1942).

[2] Wilson, R. E., *Mt. Wilson Contr.*, 669 (1942); *Ap. J.*, **96**, 371–381 (1942).

[3] Wilson, R. E., and P. W. Merrill, *Mt. Wilson Contr.*, 658 (1942); *Ap. J.*, **95**, 248–267 (1942).

[4] Van de Kamp, P., *Pop. Ast.*, **58**, 404 (1940).

[5] Our work on this highly luminous group of long-period variables was first reported at the Ottawa meeting of the American Astronomical Society, in June, 1949; Abstract, *Ast. Jour.*, **54**, 194 (1949).

[6] Kukarkin, B. W., and P. P. Parenago, *General Catalogue of Variable Stars*, Moscow (1948).

[7] *Harv. Ann.*, **60**, no. 4, 87–97 (1908).

[8] *Harv. Circ.*, 366 (1931); *Ibid.*, 439 (1940); *Ibid.*, 444 (1942); *Harv. Obs. Bull.*, **919**, 6–8 (1949); *Proc. Am. Phil. Soc.*, **92**, 310–323 (1948); Harvard Reprint Series II—25 (1948); and nine to be published in a forthcoming *Harvard Observatory Bulletin*.

PERIODIC SELECTION IN ESCHERICHIA COLI*,†

BY K. C. ATWOOD, LILLIAN K. SCHNEIDER AND FRANCIS J. RYAN

DEPARTMENT OF ZOOLOGY, COLUMBIA UNIVERSITY

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Spontaneous mutations in bacteria are generally reversible. Consequently, in a medium where prototrophy confers no selective advantage, various auxotrophs (and other mutant types) should accumulate until for each type the number is such that the loss by mutation to prototrophy should equal the gain by mutation to the corresponding auxotrophic condition. That is to say, mutational equilibria should be established.

Nevertheless it is common experience that prototrophic bacterial populations continuously maintained on complex media remain essentially prototrophic and contain at any time a very small proportion of auxotrophs. Conversely, under the same conditions, populations started from an auxotroph remain essentially auxotrophic and contain at any time a very small proportion of prototrophs. The forward and reverse mutation rates for any pair of alleles can lead to only one equilibrium, yet everyday experience shows us that two stable equilibria exist for each known mutant in bacteria. We may conclude therefore that for each mutant at least one of the equilibrium ratios cannot be attributed to a balance of mutation rates.

The above considerations serve to emphasize the ubiquitous role of selection in controlling the stability of bacterial populations which would otherwise become extremely heterogeneous as a result of mutation pressure. Perhaps it is not too surprising then, that we have found as a result of experiments to be described below that there exists in *Escherichia coli* a mechanism for stabilization of the major component of the population by the simultaneous suppression of all mutants.

Materials and Methods.—The four initial stocks of *E. coli* (strain 15) used in these experiments were histidine requiring, lactose non-fermenting (*h*− *lac*−); histidine independent, lactose non-fermenting (*h*+ *lac*−); histidine requiring, lactose fermenting (*h*− *lac*+); and histidine independent, lactose fermenting (*h*+ *lac*+). Stocks were kept on nutrient agar slants transferred every three months, grown for 36 hours at 37°C., then kept refrigerated between transfers. The main experimental procedure was the serial transfer of populations comprised of various initial ratios of the above stocks through Gray and Tatum synthetic medium containing 25γ of L-histidine monohydrochloride per cc. (*H*+). The growth rate and final level of growth of *h*− and *h*+ are indistinguishable on *H*+.[1] The serial transfer experiments (STs) consisted in transferring 0.5 cc. of cultures in stationary phase to 50 cc. of the above medium in a 125-cc. Erlenmeyer

flask. This was incubated for 12 hours at 37°C., then used as the inoculum for the next flask. The final level of growth in each flask was 2.5×10^{10}

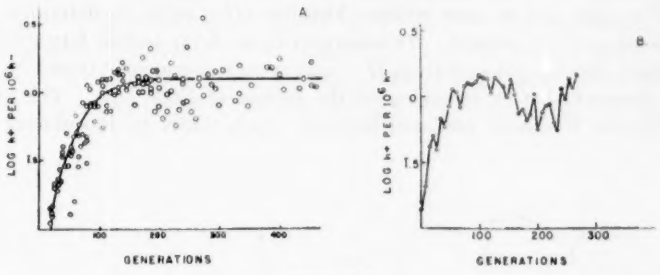

FIGURE 1

Increase of $h+$ to equilibrium. *A*, composite of eight ST experiments. *B*, data from a single ST.

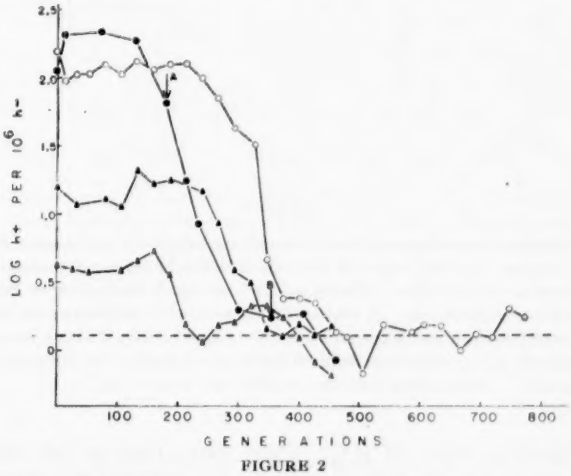

FIGURE 2

Decrease of $h+$ in four experiments started with various ratios above equilibrium. Arrow *A* indicates point of isolation of components used in reconstructions shown in figure 4 (*A*), and arrow *B* of those used in figure 4 (*B*). Broken line indicates equilibrium level.

bacteria. Thus each flask was started with an inoculum of 2.5×10^8 bacteria which grew for 6.64 generations, then remained in the stationary phase some

five hours until the next transfer. From time to time samples were taken from fully grown cultures to determine their composition. These samples were washed once by centrifugation, then plated in appropriate dilutions in $H+$ agar and in agar without histidine ($H-$ agar) to determine the proportion of $h+$ present. In experiments involving lactose fermentation markers, the $h+$ colonies from $H-$ agar were subsequently tested for lactose fermentation by streaking on the surface of Endo agar. The initial inocula for STs were prepared from the stock slants by inoculating test

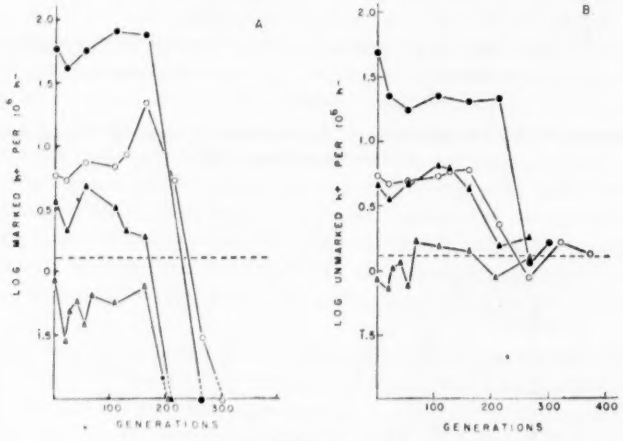

FIGURE 3

Simultaneous behavior of marked and unmarked $h+$ in the same STs. *A*, various starting ratios of $h+$ which differ in lactose fermentation character from the $h-$. Points on the base line indicate time at which no $h+$ were detected. *B*, various starting ratios of $h+$ having the same lactose fermentation character as the $h-$. Solid circles in *A* are samples from the *same* cultures as solid circles in *B*. Likewise for open circles, triangles. Broken line indicates equilibrium level.

tubes containing 10 cc. of $H+$. These were grown to full turbidity at 37°C. and mixed in the desired proportions. Where it was desired to begin the ST with no $h+$ present, the inoculum was diluted to contain about 10^4 $h-$. When STs were to be started from $h-$ and $h+$ isolated from previous STs, ten colonies from $H+$ and $H-$ agar, respectively, were isolated to nutrient agar slants and an inoculum from each was prepared as above. The ten $h-$ inocula were mixed in equal proportions, the ten $h+$ similarly mixed, and finally the desired starting ratio of $h+$ to $h-$ was constructed from the two mixtures.

Results.—In STs started from inocula containing only $h-$, the $h+$ increased until after about 15 to 20 transfers (100 to 140 generations) an equilibrium ratio of about 1 $h+$ per 10^6 $h-$ was obtained as shown in figure 1(*A*). When STs were started with various initial $h+/h-$ ratios ranging from less than 1 $h+$ to about 100 $h+$ per 10^6 $h-$, the equilibrium was reached eventually, as shown in figure 2. A curious feature of $h+/h-$ ratios above equilibrium is the plateau region of some 30 transfers which always intervenes between the start of the experiments and the fall to

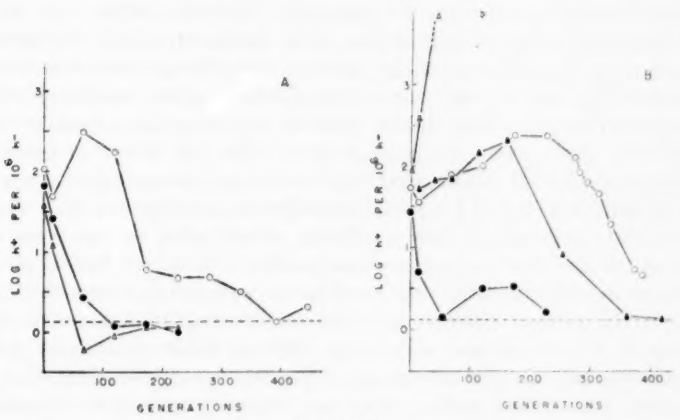

FIGURE 4

A, solid circles: $h+$ and $h-$ isolated at time indicated by arrow *A* in figure 2, and reconstructed at high ratio $h+/h-$. Triangles: Reconstruction of $h-$ isolated from arrow *A* and original $h+$. Open circles: Reconstruction of $h+$ isolated from arrow *A* and original $h-$.

B, open circles: $h+$ and $h-$ isolated at time indicated by arrow *B* in figure 2, and reconstructed at high $h+/h-$ ratio. Solid triangles: Repetition of same experiment. Solid circles: Reconstruction of $h-$ isolated from arrow *B* and original $h+$. Open triangles: Reconstruction of $h+$ isolated from arrow *B* and original $h-$.

equilibrium.[2] In addition to the data shown in figure 2, forty-eight similar experiments yielded the same results.

A similar series of STs was set up with various starting ratios of $h+$ $lac-/h-lac+$ and also of $h+lac+/h-lac-$. These experiments show the effects of selection alone, since in each case the $h+$ arising from $h-$ by mutation can be distinguished by its lactose fermentation character from the $h+$ added at the beginning of the experiment. The lactose fermentation markers were used reciprocally in different experiments to rule

out the possibility that they confer, in themselves, any selective advantage or disadvantage. In all cases the equilibrium ratio is reached by the unmarked $h+$ which arise by mutation from $h-$ (Fig. 3(B)), while the marked (added) $h+$ remain essentially constant for about 30 transfers, then rapidly disappear as shown in figure 3(A). In addition to these data thirty-six similar experiments gave the same results. Thus, both marked and unmarked $h+$ have the same duration of temporary stability.

The results of experiments with high $h+/h-$ ratios and with marked $h+$ suggested that the $h-$ had, at the time of the rapid decrease of $h+$, a selective advantage over the $h+$ present at this time. Since this interpretation seemed, superficially at least, to be contradicted by the concurrent maintenance of equilibrium of the unmarked $h+$, it was tested directly by reconstructing high $h+/h-$ ratios from the $h-$ and $h+$ components isolated from plates at a time shortly after the beginning of the rapid decline of $h+/h-$ ratio (arrow A) in experiments like that shown in figure 2. These reconstructed ratios showed in every case (eleven experiments in all) an immediate fall of $h+$ ratio to equilibrium as in figure 4(A). These experiments demonstrate that a selective advantage of $h-$ over $h+$ existed at the time the components were isolated. However, further reconstruction experiments where the components were isolated from the same ST as in the previous experiment, but shortly after equilibrium was reached (arrow B, Fig. 2), showed an entirely different result as in figure 4(B). At this time no selective advantage of $h-$ was manifested until after 30 transfers, more or less, during which time the proportion of $h+$ remained essentially constant. Finally, there was another rapid decline of $h+$ to equilibrium, just as in the previous experiments shown in figure 2. Thus we see that during the fall of $h+/h-$ ratio to equilibrium the $h-$ appears to have a selective advantage over $h+$, but when equilibrium is reached both the $h-$ and $h+$ components appear to have, temporarily, no difference in selective advantage.

Now, the question arises whether the $h+$ at equilibrium in these experiments has gained in selective advantage to match that of $h-$, or the $h-$ has somehow lost its ability to overwhelm $h+$. To decide this, $h+$ was isolated from an ST in which an initially high $h+/h-$ ratio had dropped to equilibrium, and was placed in competition with the $h-$ which had been originally used to start the ST. In experiments of this type (eight in all) the $h+$ always rapidly replaced $h-$ until the whole culture became prototrophic. Thus we see that the selective advantage which appears first in the $h-$ component in these experiments is very soon afterward possessed by both components (Fig. 4(B)).

These reconstruction experiments have been done for three STs in sequence. To make this clear we can designate the initial com-

ponents of the first ST as h_0- and h_0+. These were started at a high $h+/-h-$ ratio and after equilibrium was reached $h-$ and $h+$ were isolated from the equilibrium population. Since they both can be distinguished from h_0- and h_0+ by their greater selective advantage they can now be designated h_1- and h_1+. This process was repeated starting with h_1- and h_1+ and we obtained h_2- and h_2+. Similarly, from these we have h_3- and h_3+. The results of placing the third of these derived components in competition with the others in various combinations are shown in figure 5. Each successive cycle has resulted in a new strain which is able to overwhelm the previous strain.

In the above experiments, the bacteria with selective advantage always appear among the $h-$ first. This is necessarily the case, if they arise by random mutation, since there were anywhere from 10^4 to 10^6 times as many $h-$ as $h+$ present at all times. When experiments were done with 1:1 ratios of $h+/h-$, the mixture became unstable after a preliminary period of stability, and $h+$ became predominant in some experiments.

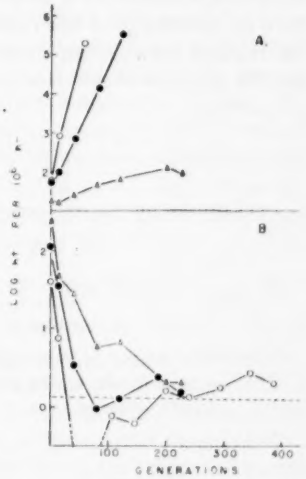

FIGURE 5

A, open circles: h_3+ and h_0-. Solid circles: h_3+ and h_1-. Triangles: h_3+ and h_2-.
B, open circles: h_3- and h_0+. Solid circles: h_3- and h_1+. Triangles: h_3- and h_2+.

Discussion.—We may regard the behavior of marked $h+$ and of $h+/h-$ ratios somewhat above equilibrium as indicative of a process which is taking place in the cultures at all times and which would pass unnoticed in the absence of such indicator conditions. This process consists of the appearance of mutants which have a selective advantage over the bacteria from which they were derived. These mutants begin to replace the bacteria which were originally present. At some time, when the number of such mutants in the population is sufficiently high, they begin to give rise to other mutants which have, in turn, a greater selective advantage. The population is again replaced by this second mutant type, and so on. Under the conditions of our experiments these mutations usually occur in the $h-$ component of the population, because $h-$ are so much more numerous. We interpret the subsequent appearance of the mutants in the smaller $h+$ component as a consequence of the mutation of $h-$ to $h+$. Thus the

mutation pattern of the above experiments may be written:

$$\begin{array}{ccccccc} h_0- & \longrightarrow & h_1- & \longrightarrow & h_2- & \longrightarrow & h_3- \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ h_0+ & & h_1+ & & h_2+ & & h_3+ \end{array}$$

As each successive type of $h-$ reaches its peak, the previous $h-$ and $h+$ are eliminated and the new $h+$ bacteria must arise by mutation from the new $h-$. According to this interpretation, the $h+/h-$ equilibrium is actually the resultant of a series of curves as shown in figure 6. Since we have no knowledge of the rates of mutation to advantageous types, or of the relation between the proportion of selectively advantageous mutants and the strength of selection pressure, we cannot tell what sort of periodic

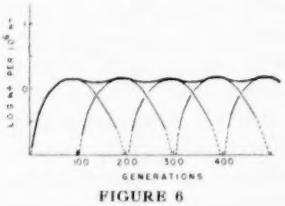

FIGURE 6

Schematic representation of equilibrium produced by periodic selection.

FIGURE 7

Hypothetical sequence of mutations which would account for perpetual periodic selection. X is the predominant type, M represents mutants which arise from X.

fluctuations in the equilibrium ratio are to be expected. Sampling error and the grouping together of data from many experiments, as in figure 1(A) conspire to mask any periodicity. In individual experiments where the population was sampled at frequent intervals, periodicity is visible as in figure 1(B). In any case, if this interpretation is correct, the equilibrium of $h+/h-$ should be maintained at that level where the number of $h+$ present is equal to the number which will be added (by mutation of $h-$ to $h+$ and subsequent growth of $h+$) in the time necessary for the initial $h+$ to disappear. Thus we see that the system would operate to produce the equivalent of selection against $h+$. Similar population shifts resulting from the appearance of more advantageous types have been reported by Stocker[6], Schneider[2], and Novick and Szilard[7].

Lieb[3] has found the mutation rate, a, from $h-$ to $h+$ to be 2.86×10^{-8}

and the mutation rate, b, from $h+$ to $h-$ to be 1.2×10^{-6}. At mutational equilibrium:

$$\frac{\hat{h}+}{\hat{h}-} = \frac{a}{b} = 2.4 \times 10^{-2} \qquad (1)$$

The observed $h+/h-$ ratio at equilibrium averages 1.3×10^{-6}. Therefore a strong selection against $h+$ must be postulated.[8] The average time necessary to eliminate the $h+$ can be estimated from experiments like those in figures 2 and 3 and appears to vary between 200 and 300 generations. Since an equilibrium is maintained, it follows that the number of $h+$ present at equilibrium must be added by mutation and growth within this time. Ryan and Schneider[4] have shown that in order to account for the rate of increase of $h+$ in small $h-$ populations where $h+$ mutants first appear, it is necessary to assume that each $h+$ produces no additional $h+$ for about two generations after it arises by mutation. They have attributed this to segregation lag, which will be fully described in their paper. If we assume a segregation lag of two generations and compute the $h+$ added per transfer from the following equation (valid for $g \geq 3$):

$$h+ = aN_0[2^{g-1} + 2^{g-2} + (g-2)2^{g-3}] \qquad (2)$$

Where $a = 2.86 \times 10^{-8}$ is the mutation rate from $h-$ to $h+$; $N_0 = 2.5 \times 10^8$ is the $h-$ inoculum of each transfer; and $g = 6.64$ is the number of generations per transfer, we find that 950 $h+$ are added per transfer, of which 9.5 on the average are included in the next inoculum. At the equilibrium of 1.3 $h+/10^6$ $h-$, 325 $h+$ are carried on the average in each inoculum. Now by the time 325 additional $h+$ are added, the original 325 must have disappeared for equilibrium to be maintained. Thus the replacement of h_1+ by h_2+, for instance, would occur with this assumption within $325/9.5 = 34$ transfers (or about 225 generations). This figure compares very favorably with the time necessary to eliminate marked $h+$ in experiments such as those in figure 3, and thus lends support to the assumption of segregation lag. It should be reemphasized that no prediction about the shape of the equilibrium curve follows from this line of reasoning. It serves only to show that the replacement phenomenon supplies selection of the right order of magnitude to account for the equilibrium.

We suppose that the periodic replacement phenomenon which we observe in these cultures would take place in the absence of the conditions for its recognition. Thus the selection against $h+$ in the presence of $h-$ may be regarded as having no relation to any intrinsic properties conferred by these genotypes. The genes in the population which are favored by periodic replacement are simply those having numerical superiority, since the selectively advantageous mutations will virtually always occur in association with these genes. When the predominant type is replaced periodi-

cally, all other mutant types which it has produced are eliminated and must arise anew by mutation (Fig. 7). Thus periodic replacement selects simultaneously against all types arising by mutation, whether or not they are selected against in any other way.

It is important to note that this does not affect the possibilities for adaptation to new environments, since if the environment selects against the predominant type, periodic replacement will be arrested until the population is transformed by the adaptive growth of the appropriate mutant. For example, if $h-$ containing a small proportion of $h+$ is placed in $H-$ medium the adaptive growth of $h+$ results.[5]

Some appreciation of the adaptive value of periodic selection can be gained from a consideration of the stability of a prototrophic population on a complex medium. There are reasons for believing that the mutation rates to auxotrophy are in general higher than the rates to prototrophy. As has been pointed out by Lieb[3] there are more ways of genetically blocking a reaction sequence than of restoring a single genetic block. In the case of $h-$ and $h+$ Lieb found the mutation rate to auxotrophy to be more than fifty times that to prototrophy. Assume that a prototrophic population which is mutating to (conservatively) 25 different auxotrophic conditions with a fiftyfold mutation pressure toward each is placed in a medium which does not select against these auxotrophs. The equilibrium value for each independent auxotrophic condition would be 49 auxotrophs to one prototroph. With only 25 such equilibria independently established, the frequency of prototrophs in this population would be about 10^{-42}. This is a total loss of ability to readapt to a simple medium. Thus, in the absence of some other equally general mechanism, periodic selection becomes a necessary condition for bacteria which remain prototrophic while growing on complex media.

In estimating the general importance of this form of selection as a stabilizer of bacterial populations, the prime consideration is whether selectively advantageous types can continue to occur indefinitely, or whether the potentiality for such mutations will be exhausted within a certain time. At this time we can present no data which give a final answer to this question. In one serial transfer experiment which lasted for 1000 generations the equilibrium was maintained throughout. If the periodic selection mechanism failed to operate, the $h+$ would increase either to mutational equilibrium or possibly to an equilibrium governed by some other mechanism of selection. Since this did not occur the conclusion is justified that the population was replaced about 4.5 times in this experiment. As we have seen, there are more general considerations which seem to demand the existence of some mechanism which acts to prevent the increase of many mutants simultaneously. While it is difficult to imagine an indefinite increase in the ability of a bacterial population to select against all ancestral

populations, it is perhaps more credible to suppose that such selective advantage might operate only on a limited number of the ancestral populations and that eventually we might arrive at a point where some ancestral type would actually have a selective advantage over the type at hand. If all of these types are accessible by mutation we may propose a mutational pattern such as shown in figure 7, in which each member can select against only a few of the previous types. This would insure the perpetual operation of the periodic selection mechanism. This scheme does not imply that the mutations are directed, since mutations to types which are temporarily disadvantageous may constantly occur without observable effect. The value of such speculation, and also the question of whether selection involves interactions between populations or is a consequence of intrinsic growth rate differences remains to be determined by further research.

Summary.—Experiments involving the serial transfer of cultures of *E. coli* for long periods of time show that, for reasons not fully understood at present, the population is periodically replaced by a new type arising in the predominant component and characterized by a selective advantage over the previous type. Thus any clone within the population has a certain maximum lifetime which limits the number of mutants which can be produced and established during its growth. Consequently, the predominant genotype does not yield to mutation pressure. This phenomenon may be regarded as a general mechanism that postpones the establishment of mutational equilibria.

† This work was supported in part by an American Cancer Society grant recommended by the Committee on Growth of the National Research Council and by a research grant from the Division of Research Grants and Fellowships of the National Institutes of Health, U. S. Public Health Service.

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MORPHOLOGICAL DIFFERENCES BETWEEN TWO SIBLING SPECIES, DROSOPHILA PSEUDOOBSCURA AND DROSOPHILA PERSIMILIS

By M. T. M. Rizki

Columbia University

Communicated by Th. Dobzhansky, December 20, 1950

The work on the population genetics of the sibling species, *Drosophila pseudoobscura* Frolova and *Drosophila persimilis* Dobzhansky and Epling has long been handicapped by the impossibility of distinguishing single individuals of these species by any morphological trait, and the consequent necessity of making cytological examinations or genetical tests. Lancefield[1] first differentiated these forms as "races or physiological species" on the basis of their genetic behavior, and called them "race A" and "race B" of *D. obscura* Fallen. Frolova and Astaurov[2] showed that Lancefield's "race A" from western United States differs from the European *D. obscura* in chromosome complement and in genital structure, and proposed for the former the name *D. pseudoobscura* Frolova. Because of the effectively complete reproductive isolation between "race A" and "race B," Dobzhansky and Epling[3] recognized the latter as a separate species, *D. persimilis*. Mather and Dobzhansky,[4] Reed, Williams and Chadwick[5] and Reed and Reed[6] made a statistical comparison of certain morphological characters in *D. pseudoobscura* and *D. persimilis*. Some average differences were found, but the overlapping of the variation curves proved so great that these differences were of no practical significance in recognition of the species.

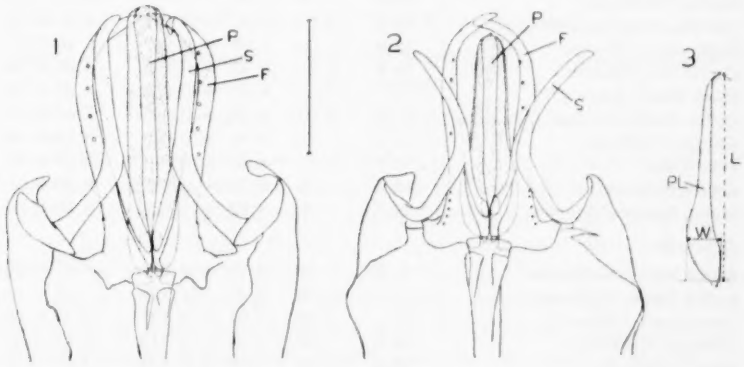
At the suggestion of Professors E. Mayr and Th. Dobzhansky, the writer undertook a detailed comparison of the male genitalia of the two species. Ten strains of *D. pseudoobscura* and ten of *D. persimilis*, derived from flies collected in various parts of the geographic distributions of each species, were used. All stocks were maintained at 16°C. The male flies were macerated in a 10% solution of KOH, washed in water, the genitalia removed, stained with carbol fuchsin and mounted in glycerin. Ten measurements from each strain were made with the aid of an ocular micrometer, one unit of which is equal to 0.0042 mm.

The structure of the genital arch and the hypandrium in the two species has been found to be alike. A clear-cut difference has, however, been noted in the dimensions and the proportions of the penis (Figs. 1 and 2). A description of the morphology and homologies of various parts of *Drosophila* male genitalia has been given by Salles.[7]

The penis of *D. pseudoobscura* is relatively long and cylindrical, whereas that of *D. persimilis* is shorter and broader at its base with a tapering appearance. The length of the penis is measured from the base of its lamina,

excluding the articular condyle, to the distal end of the lamina. Care was taken not to rupture the membrane at the tip of the penis. The width was measured at the greatest curvature of the penis' lamina with respect to the mid antero-posterior axis of the penis (Fig. 3). The penis index is computed as the ratio: length of the penis/width of lamina. The results of the measurements are summarized in table 1.

There is no significant difference among the geographical strains of *D. pseudoobscura*, nor among the strains of *D. persimilis*. However, the difference in the index of the two species is highly significant. The mean index for *D. pseudoobscura* is 8.61 ± 0.31; the mean index for *D. persimilis* is 6.48 ± 0.51. Considering the extremes in the two species, we may conclude that any individual with a penis index equal to or above 7.64 is



FIGURES 1 AND 2. Camera lucida drawings of the penis and accessory parts in *Drosophila pseudoobscura* and *D. persimilis*, respectively: *F*, forcep; *P*, penis; *S*, stylus. FIGURE 3. The measurements taken on the lamina of the penis: *L*, length; *PL*, penis lamina; *W*, width. The scale represents 100 micra.

pseudoobscura; a penis index equal to or below 7.60 indicates *persimilis*. One *pseudoobscura* male out of 100 measurements had an index 7.64. This extreme was obtained from different absolute units than the 7.60 index of *persimilis*. Considering this situation, we should employ both criteria for classification. On the basis of the shape of the penis and also the index we are definitely able to identify males of the two species.

The question that now arises is whether such morphological differences as those described are of any value in understanding sexual isolation between the species. Mayr[8] believes that some functional difficulties are encountered in heterospecific matings such that "incomplete copulation" is the result. Our findings show definite differences in the size and shape of the penis of the two species; however, we can not assign the functional

difficulties to this structure alone. In addition, as indicated by the points of articulations, the styli have a functional relation with the penis. It seems hopeful that further studies on the mechanism of the genitalia of *D. persimilis* and *D. pseudoobscura* may reveal some coordinated movements of various parts constituting a specific sequence which may differ in the two species.

TABLE 1

MEASUREMENTS AND PENIS INDICES OF DIFFERENT STRAINS OF *D. Persimilis* AND *D. Pseudoobscura*

STRAIN	MEAN LENGTH	MEAN WIDTH	INDEX MEAN	INDEX LIMITS
Pseudoobscura				
Atitlan, Guatemala	42.9	5.00	8.58 ± 0.057	8.20–8.80
Chichicastenango, Guatemala	44.5	5.00	8.90 ± 0.035	8.80–9.00
Amecameca, Mexico	43.2	5.00	8.64 ± 0.057	8.40–9.00
Cuernavaca, Mexico	43.6	4.95	8.82 ± 0.120	8.60–9.78
Black Mesa, Arizona	42.1	4.95	8.51 ± 0.066	8.20–8.89
Coffee Creek, California	41.3	5.05	8.18 ± 0.095	7.64–8.60
Alturas, California	44.2	5.15	8.59 ± 0.089	8.18–9.00
Zion, Utah	42.5	5.00	8.50 ± 0.054	8.20–8.80
Aspen, Colorado	42.5	4.95	8.59 ± 0.082	8.20–9.11
Mara, British Columbia	44.6	5.00	8.92 ± 0.066	8.60–9.20
Persimilis				
Aspen Valley, California	37.9	6.00	6.32 ± 0.029	6.17–6.50
Coffee Creek, California	39.2	6.70	5.91 ± 0.123	5.57–6.13
Deer Creek, California	38.5	6.45	5.91 ± 0.146	5.43–6.50
Nahogui, California	38.6	5.65	6.85 ± 0.114	6.33–7.40
Orick, California	36.9	5.90	6.28 ± 0.136	5.83–7.20
Porcupine Flat, California	37.3	5.45	6.85 ± 0.041	6.73–7.20
St. Helena, California	37.7	5.35	7.06 ± 0.123	6.55–7.60
Sequoia Park, California	38.5	5.85	6.59 ± 0.086	6.33–7.09
Hope, British Columbia	37.5	5.65	6.67 ± 0.165	6.00–7.60
Quesnell, British Columbia	38.4	6.00	6.40 ± 0.117	6.17–6.50
Mean penis index for *D. pseudoobscura*			8.61 ± 0.03	
Mean penis index for *D. persimilis*			6.48 ± 0.05	

NOTE: Mean length and mean width are in ocular units. The mean penis index of each strain is given with its standard error.

Summary.—Because of a complete reproductive isolation, *Drosophila pseudoobscura* and *D. persimilis* are considered separate species. However, no morphological differences between them were known. It is shown that the males of these sibling species can easily be distinguished by the shape of their genitalia.

Acknowledgments.—The writer is deeply indebted to Professor Th. Dobzhansky for his interest and guidance throughout the course of this study.

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A CONDITION FOR A REAL LATTICE TO DEFINE A ZETA FUNCTION

By N. C. ANKENY* AND C. A. ROGERS

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, AND LONDON COLLEGE, LONDON UNIVERSITY

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In Theorem 1 we shall prove the following: If the product of n linear forms in n variables, whose coefficients are real and whose determinant is positive, assumes only a finite number of different values in a finite interval, and assumes the value 0 only if all the variables are 0, then one of the linear forms arises from a ring in an algebraic number field, and the other $n - 1$ linear forms are the $n - 1$ different conjugates of the first linear form. Hence, the product is essentially the norm of all numbers in an order of an algebraic number field, which clearly takes on only a finite number of values in any finite interval.

This problem arose in connection with a forthcoming paper of S. Bochner,[1] "Some Properties of Modular Relations." We show that if any real lattice gives rise to a zeta function, the lattice comes from an order in an algebraic number field.[2] One can expand this to prove that the zeta function will satisfy a certain functional equation.[3] To define a zeta function from any real lattice it is clear that one must have certain properties of discreteness on the product of the forms to insure convergence. However, in the proof of Theorem 1, discreteness of the product is needed for only a finite interval. We thank Professor Bochner for letting us see in advance a copy of his paper.

Definition: Let x_i be real linear forms in u_j, i.e.,

$$x^{(i)} = u_1 w_1^{(i)} + u_2 w_2^{(i)} + \ldots + u_n w_n^{(i)} \qquad [i = 1, 2, \ldots, n], \quad (1)$$

where $w_j^{(i)}$ is real and the determinant $\Delta = |w_i^{(j)}|$ is positive.

Denote by

$$P(X) = x^{(1)}x^{(2)} \ldots x^{(n)} = \prod_{i=1}^{n} (u_1 w_1^{(i)} + \ldots + u_n w_n^{(i)}). \quad (2)$$

Let Λ denote the lattice consisting of all points in (1) where $u_1, \ldots, u_n$ take on all integer values.

THEOREM 1. *Suppose that there is no point X of Λ other than the origin 0 with $P(X) = 0$ and suppose that $P(X)$ assumes only a finite number of different values α, with $-\Delta \leqq \alpha \leqq \Delta$. Then the product $P(X)$ can be expressed in the form*

$$P(X) = w \prod_{i=1}^{n} (u_1 w_1^{(i)} + \ldots + u_n w_n^{(i)}),$$

where $w_1^{(1)}, \ldots, w_n^{(1)}$ are algebraic integers in a field of degree n and $w_1^{(i)}, \ldots, w_n^{(i)}$ $(i = 2, 3, \ldots, n)$ are their $(n-1)$ different algebraic conjugates.

I. *Proof:* Let $t_1, \ldots, t_{n-1}$ be $n-1$ real numbers which are linearly independent over the rational numbers. Define t_n by

$$t_1 + t_2 + \ldots + t_n = 0.$$

Write $\Delta = \delta^n$. Then by Minkowski's fundamental theorem on inequalities for linear equations, there is a point X of Λ satisfying

$$|x^{(i)}| \leqq \delta\lambda^{t_i}, \qquad i = 1, \ldots, n$$

for all positive numbers λ. For such a point we have

$$-\Delta \leqq P(X) \leqq \Delta.$$

By hypothesis, $P(X)$ can take only a finite number of values for such a point X. Thus, there is a number $\alpha \neq 0$ with the property that for suitable arbitrarily large values of λ there is a point X of Λ satisfying

$$P(X) = \alpha, \quad (3)$$

and

$$|x^{(i)}| \leqq \delta\lambda^{t_i}, \qquad i = 1, \ldots, n. \quad (4)$$

Let Γ be the set of points X of Λ satisfying $P(X) = \alpha$. Let $U_1, U_2, \ldots, U_h$ be the different products which can be obtained by multiplying between 1 and n of the variables $u_1, \ldots, u_n$. Then we can write

$$P(X) = \Omega_1 U_1 + \ldots + \Omega_h U_h.$$

We know that for every point X of Γ

$$P(X) = \Omega_1 U_1 + \ldots + \Omega_h U_h = \alpha$$

and that $U_1, \ldots, U_h$ are integers as the u's are integers. For each point X of Γ we regard the result

$$\Omega_1 U_1 \ldots \Omega_h U_h = \alpha \quad (5)$$

as an equation for $\Omega_1, \ldots, \Omega_h$.

II. Our first object is to prove that this system of equations (5) determines $\Omega_1, \ldots, \Omega_h$ uniquely. We suppose that the equations have a second solution $\Omega_1 + \Theta_1, \ldots, \Omega_h + \Theta_h$. Then we have

$$\Theta_1 U_1 + \ldots + \Theta_h U_h = 0$$

for all points X of Γ. As $\Delta \neq 0$, the $u_1, \ldots, u_n$ are linear forms in $x^{(1)}, \ldots, x^{(n)}$. So we can express

$$\Theta_1 U_1 + \ldots + \Theta_h U_h$$

as a polynomial in $x^{(1)}, \ldots, x^{(n)}$. Let $Y_1, Y_2, \ldots$ be the monomial expressions of $x^{(1)}, \ldots, x^{(n)}$ arranged in some order. Then $\Theta_1 U_1 + \ldots + \Theta_h U_h$ will be a linear form in $Y_1, Y_2, \ldots$. We may choose polynomials $p_1(x)$, $p_2(x), \ldots$ so that

$$\Theta_1 U_1 + \ldots + \Theta_h U_h = \sum_\rho p_\rho(x^{(1)} \ldots x^{(n)}) Y_\rho,$$

where $p(x)$ vanishes identically whenever Y_ρ is divisible by $x^{(1)}x^{(2)} \ldots x^{(n)}$. We may suppose that $Y_1, Y_2, \ldots$ are ordered so that

$$\begin{aligned} &p_\rho(x) \text{ does not vanish identically for } \rho \leqq r, \\ &p_\rho(x) \text{ vanishes identically for } \rho > r. \end{aligned}$$

Then we have

$$\Theta_1 U_1 + \ldots + \Theta_h U_h = \sum_{\rho=1}^{r} p_\rho(P(X)) Y_\rho$$

identically and for all X of Γ

$$\sum_{\rho=1}^{r} p_\rho(P(X)) Y_\rho = 0. \tag{6}$$

We take a sequence of values of λ tending to $+\infty$ and consider a corresponding sequence of points X of Γ. We have

$$\left.\begin{aligned} P(X) = \prod_{i=1}^{n} x^{(i)} &= \alpha \\ |x^{(i)}| \leqq \delta\lambda^{t_i}, \qquad i &= 1, \ldots, n. \end{aligned}\right\} \tag{7}$$

So

$$|x^{(i)}| \geqq \frac{|\alpha|}{\delta^{n-1}} \delta^{t_i}, \qquad i = 1, \ldots, n.$$

If

$$Y_\rho = (x^{(1)})^{\beta_{1\rho}} (x^{(2)})^{\beta_{2\rho}} \ldots (x^{(n)})^{\beta_{n\rho}},$$

where $\beta_{1\rho}, \ldots, \beta_{n\rho}$ are integers, we see that as the exact order of Y_ρ is

$$0(\lambda^{t_1\beta_{1\rho} + \ldots + t_n\beta_{n\rho}}).$$

If two of the monomials $Y_1, \ldots, Y_r$, say Y_ρ and Y_σ, had the same order of magnitude as $\lambda \to \infty$, we would have

$$t_1\beta_{1\rho} + \ldots + t_n\beta_{n\rho} = t_1\beta_{1\sigma} + \ldots + t_n\beta_{n\sigma}$$

so that

$$t_1(\beta_{1\rho} - \beta_{1\sigma}) + \ldots + t_n(\beta_{n\rho} - \beta_{n\sigma}) = 0$$

and by our choice of $t_1, \ldots, t_n$ we would have

$$\beta_{1\rho} - \beta_{1\sigma} = \beta_{2\rho} - \beta_{2\sigma} = \ldots = \beta_{n\rho} - \beta_{n\sigma}.$$

But this is impossible unless $\sigma = \rho$ since neither Y_ρ nor Y_σ is divisible by $x^{(1)} \ldots x^{(n)}$. Thus no two of $Y_1, \ldots, Y_r$ have the same order of magnitude as $\lambda \to \infty$ and X runs through a sequence of points of Γ satisfying (7). Since these points all satisfy (6) we see that

$$p_\rho(\alpha) = 0.$$

Hence, the polynomials $p_1(X), \ldots, p_r(X)$ are divisible by the polynomial $(X - \alpha)$. Comparing the degrees in the identity

$$\Theta_1 U_1 + \ldots + \Theta_h U_h = \sum_{\rho=1}^{r} p_\rho(P(X)) Y_\rho,$$

we see that $r = 1$, $Y_1 = 1$ and

$$\Theta_1 U_1 + \ldots + \Theta_h U_h = a(P(X) - \alpha)$$

for some constant a. But the expression on the left has no constant term and α is $\neq 0$. Hence, $a = 0$, and

$$\Theta_1 = \Theta_2 = \ldots = \Theta_h = 0.$$

Thus the equations (5) determine $\Omega_1, \ldots, \Omega_h$ uniquely.

III. As the numbers $U_1, \ldots, U_h$ are rational integers, we have by Section II that the Ω's are rational multiples of α. Therefore,

$$P(X) = \prod_{i=1}^{n} (u_1 w_1^{(i)} + \ldots + u_n w_n^{(i)}) \qquad (8)$$

when expressed as a polynomial in $u_1, \ldots, u_n$ has all coefficients as rational multiples α. Divide (8) by $(w_1^{(1)} w_1^{(2)} \ldots w_1^{(n)})$ giving

$$wP(X) = \prod_{i=1}^{n} (u_1 + u_2 v_2^{(i)} + \ldots + u_n v_n^{(i)}) \qquad (9)$$

when expressed as a polynomial in $u_1, \ldots, u_n$ has all its coefficients rational mutiples of

$$\frac{\alpha}{(w_1^{(1)}w_1^{(2)} \ldots w^{(n)})^*}$$

However, the coefficient of u_1^n is 1. Hence all the coefficients are rational numbers.

Letting $u_2 = 1, u_3 = u_4 = \ldots = u_n = 0$ we see $\nu_2^{(i)}$ for $(i = 1, 2, \ldots, n)$ are all algebraic numbers. Similarly, we have $\nu_j^{(i)}$ for $i, j = 1, 2, \ldots, n$ are all algebraic numbers.

Let $K = R(\nu_2^{(1)}, \nu_3^{(1)}, \ldots, \nu_n^{(1)})$, where R is the rational numbers (i.e., K is the least algebraic extension of R containing $\nu_2^{(1)}, \ldots, \nu_n^{(1)}$). As $u_1 + \nu_2^{(1)}u_2 + \ldots + \nu_n^{(1)}u_n \neq 0$ unless $u_1 = u_2 = \ldots = u_n = 0$, we see that $(K:R) = n' \geqq n$. Hence, consider the n' different isomorphisms of K leaving R fixed. As they leave R fixed they leave the coefficients of the u's in the expansion of (9) fixed. By Hilbert's theorem on the uniqueness of the irreducible factors of a polynomial in n variables, we see that the n' isomorphisms of K must merely permute the linear factors on the right side of (9). As all isomorphisms of K act differently on $u_1 + u_2\nu_2^{(1)} + \ldots + u_n\nu_n^{(1)}$, we see that $n' = n$, and the linear factors on the right-hand side of (9) are the $n - 1$ different conjugates. This proves Theorem 1.

IV. It is of some interest to note that to prove Theorem 1 we do not need the full hypothesis of discreteness. Actually we only need to note the fact that there exists one $\alpha \not\equiv 0$ which satisfies (3) and (4) for sufficiently large values of λ. This modification has application to the critical lattice of n dimensional paraboloids. However, in this present note, we cannot take up this problem in any detail.

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THE JACOBSON RADICAL OF A SEMIRING

BY SAMUEL BOURNE

INSTITUTE OF ADVANCED STUDY, PRINCETON, NEW JERSEY*

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1. Introduction.—A semiring is a system consisting of a set S together with two binary operations, called addition and multiplication, which forms a semigroup relative to addition, a semigroup relative to multiplication, and the right and left distributive laws hold. This system was first introduced by Vandiver.[1] He also gave examples[2] of semirings which cannot

be imbedded in a ring. Semirings arise naturally when we consider the set of endomorphisms of a commutative additive semigroup.[3]

Our purpose is to generalize the concept of the Jacobson radical of a ring[4] to arbitrary semirings. In section 2 we define the concept of an ideal in a semiring S and develop the corresponding homomorphism theorem for semirings. In section 3 we extend the definition of the Jacobson radical to arbitrary semirings, and in section 4 we obtain some properties of the Jacobson radical of a semiring. We conclude with a consideration, in section 5, of the Jacobson radical of matrix semiring S_n.

This paper has profited greatly from discussion with C. A. Rogers, a colleague of mine at the Institute.

2. The Homomorphism Theorem.—We shall assume that the additive semigroup of S is commutative and that S possesses a zero element. The latter assumption is not vital in the sense that if S lacked a zero element, we can easily adjoin one to S.

Definition 1: An ideal of S is a subset I of S containing zero such that if i_1 and i_2 are in I, then $i_1 + i_2$ is in I, and if i is in I, and s is any element of S, then is and si are in I.

We shall say that s_1 is equivalent to s_2 modulo the ideal I, if there exist elements i_1 and i_2 of the ideal I such that $s_1 + i_1 = s_2 + i_2$. This definition is a translation to the additive notation of one given by Dubreil[5] for a multiplicative semigroup. This relationship is obviously an equivalence. The set of elements $\bar{s}$ equivalent to each other modulo the ideal I is called a coset of the ideal I. Relative to the usual definitions of an addition and multiplication the cosets of an ideal I of a semiring form a semiring, called the difference semiring $S - I$. The correspondence $s \to \bar{s}$ defines a homomorphism of S into the difference semiring $\bar{S} = S - I$. Conversely, if the semiring S is homomorphic to the semiring S', then the elements of S mapped into $0'$ of S' form an ideal I, and S' is isomorphic to $\bar{S} = S - I$, via the one-to-one correspondence $s' \longleftrightarrow \bar{s}$. We have proved

THEOREM 1. *If the semiring S is homomorphic to the semiring S', then S' is isomorphic to the difference semiring $S - I$, where I is the ideal of elements mapped into 0'. Conversely, if I is an ideal of S, then S is homomorphic to the difference semiring $S - I$.*

3. The Jacobson Radical of a Semiring.—Definition 2: The element r of the semiring S is said to be right semiregular if there exist element r' and r'' in S such that $r + r' + rr' = r'' + rr''$.

We notice that in the case S is a ring, this definition reduces to the one usually given for right quasi-regularity in a ring.

THEOREM 2. *A necessary and sufficient condition that the element r of S be right semiregular is that for any element s in S there exist elements s' and s'' in S such that $s + s' + rs' = s'' + rs''$.*

Proof: If r is right semiregular, then $r + r' + rr' = r'' + rr''$. There-

fore, $s + r''s + rr''s = s + r's + r(s + r's)$. On letting $s' = r''s$ and $s'' = s + r's$, we obtain $s + s' + rs' = s'' + rs''$.

Conversely, we suppose that $s + s' + rs' = s'' + rs''$, for any s in S. In particular, $r + s' + rs' = s'' + rs''$ and r is right semiregular.

Definition 3: The right ideal I is said to be right semiregular, if for every pair of elements i_1, i_2 in I there exist elements j_1 and j_2 in I such that $i_1 + j_1 + i_1j_1 + i_2j_2 = i_2 + j_2 + i_1j_2 + i_2i_1$.

The elements j_1 and j_2 are not unique, for this condition implies that $i_1 + j_1 + i_1j_1 + i_2j_2 + j = i_2 + j_2 + i_1j_2 + i_2j_1 + j$, where j is any element in I, and $i_1 + (j_1 + j) + i_1(j_1 + j) + i_2(j_2 + j) = i_2 + (j_2 + j) + i_1(j_2 + j) + i_2(j_1 + j)$. In particular, on substituting $i_2 = 0$, we obtain that i_1 is right semiregular.

LEMMA 1. *If I and I^* are right semiregular ideals, then $I + I^*$ is a right semiregular ideal.*

Proof: Since I is right semiregular, for every pair of elements i_1, i_2 in I there exist elements j_1 and j_2 in I such that $i_1 + j_1 + i_1j_1 + i_2j_2 = i_2 + j_2 + i_1j_2 + i_2j_1$. If i_1^* and i_2^* are in I^*, then $i_1^* + i_1^*j_1 + i_2^*j_2$ and $i_2^* + i_2^*j_1 + i_1^*j_2$ are in I^*.

Since I^* is right semiregular, there exist elements j_1^* and j_2^* in I^* such that $(i_1^* + i_1^*j_1 + i_2^*j_2) + j_1^* + (i_1^* + i_1^*j_1 + i_2^*j_2)j_1^* + (i_2^* + i_2^*j_1 + i_1^*j_2)j_2^* = (i_2^* + i_2^*j_1 + i_1^*j_2) + j_2^* + (i_1^* + i_1^*j_1 + i_2^*j_2)j_2^* + (i_2^* + i_2^*j_1 + i_1^*j_2)j_1^*$.

Therefore, $(i_1 + i_1^*) + (j_1 + j_1^* + j_1j_1^* + j_2j_2^*) + (i_1 + i_1^*)(j_1 + j_1^* + j_1j_1^* + j_2j_2^*) + (i_2 + i_2^*)(j_2 + j_2^* + j_1j_2^* + j_2j_1^*) = (i_1 + j_1 + i_1j_1 + i_2j_2) + [(i_1^* + i_1^*j_1 + i_2^*j_2) + j_1^* + (i_1^* + i_1^*j_1 + i_2^*j_2)j_1^* + (i_2^* + i_2^*j_1 + i_1^*j_2)j_2^*] + (i_1 + j_1 + i_1j_1 + i_2j_2)j_1^* + (i_2 + j_2 + i_1j_2 + i_2j_1)j_2^* = (i_2 + j_2 + i_2j_2 + i_2j_1) + [(i_2^* + i_2^*j_1 + i_1^*j_2) + j_2^* + (i_1^* + i_1^*j_1 + i_2^*j_2)j_2^* + (i_2^* + i_2^*j_1 + i_1^*j_2)j_1^*] + (i_2 + j_2 + i_1j_2 + i_2j_1)j_1^* + (i_1 + j_1 + i_1j_1 + i_2j_2)j_2^* = (i_2 + i_2^*) + (j_2 + j_2^* + j_2j_1^* + j_1j_2^*) + (i_1 + i_1^*)(j_2 + j_2^* + j_2j_1^* + j_1j_2^*) + (i_2 + i_2^*)(j_1 + j_1^* + j_1j_1^* + j_2j_2^*)$.

Since $j_1 + j_1^* + j_1j_1^* + j_2j_2^*$ and $j_2 + j_2^* + j_1j_2^* + j_2j_1^*$ are in $I + I^*$, this equation implies that $I + I^*$ is right semiregular.

THEOREM 3. *The sum R of all the right semiregular ideals of a semiring S is a right semiregular two-sided ideal.*

Proof: Lemma 1 implies that R is right semiregular. If r_1 and r_2 are in R, then r_1s and r_2s are in R, for any s in S. Hence there exist elements r_3 and r_4 in R such that $r_1s + r_3 + r_1sr_3 + r_2sr_4 = r_2s + r_4 + r_1sr_4 + r_2sr_3$. Whence, $sr_1sr_1 + sr_3r_1 + sr_1sr_3r_1 + sr_2sr_4r_1 = sr_2sr_1 + sr_4r_1 + sr_1sr_4r_1 + sr_2sr_3r_1$ and $sr_2sr_2 + sr_4r_2 + sr_1sr_4r_2 + sr_2sr_3r_2 = sr_1sr_2 + sr_3sr_2 + sr_1sr_3r_2 + sr_2sr_4r_2$. Adding the last two equations and adding $sr_1 + sr_2$ to both sides of the resulting equation we obtain that $sr_2 + (sr_1 + sr_3r_1 + sr_4r_2) + sr_1(sr_1 + sr_3r_1 + sr_4r_2) + sr_2(sr_4r_1 + sr_2 + sr_3r_2) = sr_1 + (sr_2 + sr_3r_2 + sr_4r_1) + sr_2(sr_3 + sr_3r_1 + sr_4r_2) + sr_1(sr_2 + sr_3r_2 + sr_4r_1)$. Since $sr_1 +$

$sr_3r_1 + sr_4r_3$ and $sr_2 + sr_3r_2 + sr_4r_1$ are in sR, this equation implies that sR is right semiregular and $sR \leq R$. Hence R is also a left ideal and R is a two-sided ideal.

Definition 4: The right Jacobson radical R of a semiring S is the sum of all the right semiregular ideals of S.

In a corresponding manner, we obtain the left Jacobson radical R' as the sum of all the left semiregular ideals of S. An ideal of S is said to be semiregular if it is both right semiregular and left semiregular.

LEMMA 2. *If i_1 and i_2 are contained in the right ideal I of S, and if $i_1 + j_1 + i_1j_1 + i_2j_2 = i_2 + j_2 + i_2j_2 + i_2j_1$ and $i_1 + k_1 + k_1i_1 + k_2i_2 = i_2 + k_2 + k_2i_1 + k_1i_2$, where j_1, j_2, k_1 and k_2 are in I, then there exists an element l in I such that $k_1 + j_2 + l = k_2 + j_1 + l$.*

Proof: $k_1 + (i_1 + j_1 + i_1j_1 + i_2j_2) + k_1(i_1 + j_1 + i_1j_1 + i_2j_2) + k_2(i_2 + j_2 + i_1j_2 + i_2j_1) = j_1 + (i_1 + k_1 + k_1i_1 + k_2i_2) + (i_1 + k_1 + k_1i_1 + k_2i_2)j_1 + (i_2 + k_2 + k_2i_1 + k_1i_2)j_2$.

Hence $k_1 + (i_2 + j_2 + i_1j_2 + i_2j_1) + k_1(i_2 + j_2 + i_1j_2 + i_2j_1) + k_2(i_1 + j_1 + i_1j_1 + i_2j_2) = j_1 + (i_2 + k_2 + k_2i_1 + k_1i_2) + (i_2 + k_2 + k_2i_1 + k_1i_2)j_1 + (i_1 + k_1 + k_1i_1 + k_2i_2)j_2$.

Therefore, $k_1 + j_2 + (i_1 + k_1 + k_1i_1 + k_2i_2)j_2 + (i_2 + k_2 + k_1i_2 + k_2i_1)j_1 + i_2 + k_1i_2 + k_2i_1 = k_2 + j_1 + (i_2 + k_2 + k_2i_1 + k_1i_2)j_1 + (i_1 + k_1 + k_1i_1 + k_2i_2)j_2 + i_2 + k_2i_1 + k_1i_2$.

We define $l = (i_2 + k_2 + k_2i_1 + k_1i_2)j_1 + (i_1 + k_1 + k_1i_1 + k_2i_2)j_2 + i_2 + k_2i_1 + k_1i_2$.

Then $k_1 + j_2 + l = k_2 + j_1 + l$, where l is in I.

LEMMA 3. *The right Jacobson radical R of a semiring S is a left semiregular ideal.*

Proof: Theorem 3 states that R is a left ideal. We let r_1 and r_2 be any two elements of R. Since R is right semiregular, there exist elements s_1 and s_2 in R such that $r_1 + s_1 + r_1s_1 + r_2s_2 = r_2 + s_2 + r_1s_2 + r_2s_1$. Therefore, $s_1 + t_1 + s_1t_1 + s_2t_2 = s_2 + t_2 + s_1t_2 + s_2t_1$, where t_1 and t_2 are in R.

By Lemma 2 there exists an element v in R such that $t_1 + r_2 + v = t_2 + r_1 + v$. Therefore, $s_1 + r_1 + s_1t_1 + s_2t_2 + t_1 + v = s_2 + t_2 + r_1 + v + s_1t_2 + s_2t_1$ or $s_1 + r_1 + s_1t_1 + s_2t_2 + t_1 + v = s_2 + r_2 + s_1t_2 + s_2t_1 + t_1 + v$. Whence, $s_1 + r_1 + s_1r_1 + s_2t_2 + s_1t_1 + s_1v + t_1 + v = s_2 + r_2 + s_1(t_2 + r_1 + v) + t_1 + v + s_2t_1$ or $s_1 + r_1 + s_1r_1 + s_2t_2 + s_1t_1 + s_1v + t_1 + v = s_2 + r_2 + s_2t_1 + s_1t_1 + s_1v + t_1 + v + s_1r_2$.

Finally, $s_1 + r_1 + s_1r_1 + s_2r_2 + u = s_1 + r_2 + s_1r_2 + s_2r_1 + u$, where u is in R. Since s_1 and s_2 are in R, this equation implies that R is a left semiregular ideal.

THEOREM 4. *The right Jacobson radical of a semiring S is equal to its left Jacobson radical R'.*

Proof: Lemma 3 implies that $R \leq R'$. From the symmetry of our results, we have that $R' \leq R$. Hence $R = R'$.

We may now refer to the Jacobson radical R of a semiring S.

4. Properties of the Jacobson Radical.—Definition 5: The semiring S is said to be semisimple if its Jacobson radical $R = (0)$.

Definition 6: The semiring S is said to be a radical semiring if its Jacobson radical $R = S$.

THEOREM 5. *If R is the Jacobson radical of the semiring S, then the difference semiring $S - R$ is semisimple.*

Proof: We let $\bar{S} = S - R$ and $\bar{R}$ be the Jacobson radical of $\bar{S}$. If $\bar{r}_1$ and $\bar{r}_2$ are in $\bar{R}$, there exist elements $\bar{s}_1$ and $\bar{s}_2$ in $\bar{R}$ such that $\bar{r}_1 + \bar{s}_1 + \bar{r}_1\bar{s}_1 + \bar{r}_2\bar{s}_2 = \bar{r}_2 + \bar{s}_2 + \bar{r}_1\bar{s}_2 + \bar{r}_2\bar{s}_1$. This implies that $r_1 + s_1 + r_1s_1 + r_2s_2 + r_3 = r_2 + s_2 + r_1s_2 + r_2s_1 + r_4$, where r_3 and r_4 are in R, and r_1, r_2, s_1 and s_2 are in the set R^* of all elements of S belonging to the cosets of $\bar{R}$. R^* is an ideal in S, since $\bar{R}$ is an ideal in $\bar{S}$. R^* contains R, since $\bar{0}$ is in $\bar{R}$.

Since r_3 and r_4 are in R, there exist elements s_3 and s_4 in R such that $r_4 + s_4 + r_4s_4 + r_3s_3 = r_3 + s_3 + r_3s_4 + r_4s_3$.

If we denote the left-hand side of this equation by (e) and its right-hand side by $(e)'$, then $(e) + s_3 + (e)s_4 + (e)'s_3 = (e)' + s_3 + (e)'s_4 + (e)s_3$ yields that $r_1 + s_1 + r_1s_1 + r_2s_2 + r_3 + s_3 + r_1s_4 + s_1s_4 + r_1s_1s_4 + r_2s_2s_4 + r_3s_4 + r_2s_3 + s_2s_3 + r_1s_2s_3 + r_2s_1s_3 + r_4s_3 = r_1 + s_1 + s_1s_4 + s_2s_3 + r_1(s_1 + s_4 + s_1s_4 + s_2s_3) + r_2(s_2 + s_2s_4 + s_3 + s_1s_3) + r_3 + s_3 + r_3s_4 + r_4s_3 = r_2 + s_2 + s_2s_4 + s_1s_3 + r_1(s_2 + s_3 + s_2s_4 + s_1s_3) + r_2(s_1 + s_4 + s_1s_4 + s_2s_3) + r_4 + s_3 + r_4s_4 + r_3s_3$.

We let $t_1 = s_1 + s_1s_4 + s_2s_3$, $t_2 = s_2 + s_2s_4 + s_1s_3$ $t_3 = r_3 + r_3s_4 + r_4s_3$ $t_4 = r_4 + r_4s_4 + r_3s_3$.

Since $r_4 + s_4 + r_4s_4 + r_3s_3 = r_3 + s_3 + r_3s_4 + r_4s_3$, we have that $t_2 + s_3 = t_4 + s_4$. On substituting we obtain that $r_1 + t_1 + r_1(t_1 + s_4) + r_2(t_2 + s_3) + t_4 + s_4 = r_2 + t_2 + r_1(t_2 + s_3) + r_2(t_1 + s_4) + t_4 + s_3$.

On adding to both sides $(r_1 + r_2)t_4$ we obtain that $r_1 + (t_1 + s_4 + t_4) + r_1(t_1 + s_4 + t_4) + r_2(t_2 + s_3 + t_4) = r_2 + (t_2 + s_3 + t_4) + r_1(t_2 + s_3 + t_4) + r_2(t_1 + s_4 + t_4)$.

Since $t_1 + s_4 + t_4$ and $t_2 + s_3 + t_4$ are in R^*, this equation states that R^* is a semiregular ideal in S. Therefore $R^* \leq R$ and $\bar{R} = (\bar{0})$.

THEOREM 6. *The Jacobson radical of a semiring S is a radical semiring.*

Proof: We let $R(S)$ denote the Jacobson radical of S and $R(R(S))$ the Jacobson radical of $R(S)$. Since $R(S)$ is semiregular in $R(S)$, then $R(S) \leq R(R(S))$. Trivially, $R(R(S)) \leq R(S)$. Whence, $R(R(S)) = R(S)$.

Thus the structure of an arbitrary semiring is reduced to the consideration of semisimple semirings and radical semirings.

If $r^n = 0$, then on letting $r' = \sum_1^{n-1} r^{2i}$ and $r'' = \sum_1^{n} r^{2i-1}$ we obtain that $r + r' + rr' = r'' + rr''$. Hence if an element is nilpotent in S, it is automatically semiregular. However, the converse of this statement is

false. For, in the semiring of non-negative rational numbers 2 is semi-regular while not nilpotent.[6]

THEOREM 7. *Every nilpotent right or left ideal of a semiring S is contained in the Jacobson radical R of S.*

Proof: We let I be a nilpotent right ideal of degree n, that is $I^n = (0)$. We wish to prove that I is a right semiregular ideal. It can be easily verified that, if we define

$$s_t = \sum_{k=1}^{2n-1} \sum_{j_1, \ldots, j_k = 1, 2} \epsilon_t(j_1, \ldots, j_k) i_{j_1} \ldots i_{j_k}, \quad t = 1, 2$$

and

$$\epsilon_1(j_1, \ldots, j_k) = \begin{cases} 1 \text{ if } j_s = 1 & \text{for an even number of } s, \\ 0 \text{ if } j_s = 1 & \text{for an odd number of s,} \end{cases} \quad s = 1, \ldots, k,$$

$$\epsilon_2 = 1 - \epsilon_1(j_1, \ldots, j_k),$$

then $i_1 + s_1 + i_1 s_1 + i_2 s_2 = i_2 + s_2 + i_2 s_1 + i_1 s_2$, where s_1 and s_2 are in I. This implies that I is a right semiregular ideal and that $I \leq R$.

Definition 7: The principal right ideal (v) is the set of elements $\{vs + vn\}$, where s is an arbitrary element of the semiring S and n a non-negative integer.

COROLLARY. *If v is an element such that $SvS \leq R$, then $v \leq R$.*

Proof: If I is the principal right ideal (v), then $I_3 \leq SIS$. However, $SvS = SIS$. Trivially, $SvS \leq SIS$. An element of SIS is of the form $\sum s_1(vs + vn)s_2 = \sum s_1 vss_2 + s_1 vns_2 = \sum s_1' vs_2'$. Therefore, $SIS \leq SvS$ and the result follows. Thus $I^3 \leq SvS \leq R$. Now $(I + R)^3 = \sum(i_1 + r_1)(i_2 + r_2)(i_3 + r_3) = \sum i_2 i_2 i_3 + \sum r$. Therefore, $(I + R)^3 \leq R$. We have then that $\bar{I} = (I + R) - R$ is a nilpoint ideal of degree 3 in the semisimple semiring $\bar{S} = S - R$. Theorem 7 states that $\bar{I} \leq \bar{R} = (\bar{0})$, by theorem 6. Therefore $I \leq R$, and, in particular, $v \in R$.

Definition 8: A semiring S is said to be regular, in the sense of von Neumann,[7] if for every element s in S there exist elements x' and x'' in S such that $sx's = s + sx''s$.

We notice that in the case S is a ring, this definition reduces to the one given by von Neumann.

THEOREM 8. *If R is the radical of a regular semiring S, then for any element r in R there exists an element w in R such that $r + w = w$.*

Proof: We let $r \in R$, then $rx'r = r + rx''r$. Since $x''r$ and $x'r$ are in R, there exist elements s_1 and s_2 in R such that $x''r + s_1 + (x''r)s_1 + (x'r)s_2 = x'r + s_2 + (x'r)s_1 + (x''r)s_2$. Therefore, $rx''r + rs_1 + rx''rs_1 + rx'rs_2 = rx'r + rs_2 + rx'rs_1 + rx''rs_2$. Whence, $rx''r + rs_1 + rx''rs_1 + rs_2 + rx''rs_2 = r + rx''r + rs_2 + rs_1 + rx''rs_1 + rx''rs_2$. We define $w = rx''r + rs_1 + rx''rs_1 + rs_2 + rx''rs_2$ and obtain that $r + w = w$, $w \in R$.

We notice that in the case S is a regular ring, this theorem implies that S is a semisimple ring.

5. The Jacobson Radical of a Matrix Semiring S_n.—LEMMA 4. *If s_{11} is right semiregular in the semiring S, then the matrix $M = (s_{ij})$, $s_{ij} = 0$, $i > 1$, is right semiregular in the matrix semiring S_n.*

Proof: Theorem 2 yields that $s + s' + s_{11}s' = s'' + s_{11}s''$. Therefore, $s_{1j} + s_{1j}' + s_{11}s_{1j}' = s_{1j}'' + s_{11}s_{1j}''$. We have, then, that $M + M' + MM' = M'' + MM''$, where $M' = (s_{ij}')$, $s_{ij}' = 0$, $i > 1$, and $M'' = (s_{ij}'')$, $s_{ij}'' = 0$, $i > 1$. Thus M is right semiregular in S_n.

LEMMA 5. *If R is the Jacobson radical of a semiring S, then the right ideal I_k of all matrices $M = (r_{ij})$, $r_{ij} \in R$, $r_{ij} = 0$, $i \neq k$, is a right semiregular ideal in S_n.*

Proof: The fact that I_k is a right ideal is obvious. We let $M_1 = (r_{ij})$ and $M_1^* = (r_{ij}^*)$ be any two matrices of I_k. We consider the pairs of elements (r_{kj}, r_{kj}^*), $j = 1, 2, \ldots, n$. Since r_{kj} and r_{kj}^* are in R, there exist elements s_{kj}, s_{kj}^* in R such that $r_{kj} + s_{kj} + r_{kj}s_{kj} + r_{kj}^*s_{kj}^* = r_{kj}^* + s_{kj}^* + r_{kj}^*s_{kj} + r_{kj}s_{kj}^*$. We define $S_1 = (s_{ij})$, $s_{ij} = 0$, $i \neq k$ and $S_1^* = (s_{ij}^*)$, $s_{ij}^* = 0$, $i \neq k$. Then, we can easily verify that $M_1 + S_1 + M_1S_1 + M_1^*S_1^* = M_1^* + S_1^* + M_1^*S_1 + M_1S_1^*$, where S_1 and S_1^* are in I_k. We have proved that I_k is a right semiregular ideal in S_n.

THEOREM 9. *If R is the Jacobson radical of a semiring S with a unit element, then R_n is the Jacobson radical $R(S_n)$ of the matrix semiring S_n.*

Proof: Now $R_n = \sum_1^n I_k$. Since Lemma 5 states that I_k is a right semiregular ideal, then R_n is a right semiregular ideal by Lemma 1. Hence, $R_n \leq R(S_n)$.

We wish to prove that $R(S_n) \leq R_n$. We let $M = (s_{ij}) \in R(S_n)$ and $T_{ij} = (t_{lm})$, $t_{lm} = 0$, $l \neq i$, $m \neq j$, $t_{ij} = 1$. Then $D = [s_{pq}, \ldots, s_{pq}] = \sum_k T_{kp}MT_{qk}$ is contained in $R(S_n)$.

We consider the totality of diagonal matrices of the form $D = [d, \ldots, d]$ which are contained in $R(S_n)$. Then the set Δ of elements, which are contained in the first place of D, forms a right ideal in S. If $D_1 = [d_1, \ldots, d_1]$ and $D_2 = [d_2, \ldots, d_2]$ are in $R(S_n)$, there exist matrices S_1 and S_2 in $R(S_n)$ such that $D_1 + S_1 + D_1S_1 + D_2S_2 = D_2 + S_2 + D_1S_2 + D_2S_1$. Using matrices of type T_{ij}, previously defined, we can modify this to read $D_1 + E_1 + D_1E_1 + D_2E_2 = D_2 + E_2 + D_1E_2 + D_2E_1$, where $E_k = [e_{11}^k, e_{11}^k, \ldots, e_{11}^k]$, $k = 1, 2$. We have then that $d_1 + e_{11}^1 + d_1e_{11}^1 + d_2e_{11}^2 = d_2 + e_{11}^2 + d_1e_{11}^2 + d_2e_{11}^1$, where e_{11}^1 and e_{11}^2 are in Δ. This implies that Δ is a right semiregular ideal and $\Delta \leq R$. This yields that $s_{pq} \in R$ and $M \leq R_n$. Therefore, $R(S_n) \leq R_n$ and $R(S_n) = R_n$.

* Present address: University of Connecticut, Storrs, Conn.

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REPRESENTATIONS OF SEMISIMPLE LIE GROUPS ON A BANACH SPACE

BY HARISH-CHANDRA

DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY

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Let G be a connected semisimple Lie group and $\mathfrak{H}$ a Banach space. By a representation of G on $\mathfrak{H}$ we mean a mapping π which assigns to every $x \in G$ a bounded linear operator $\pi(x)$ on $\mathfrak{H}$ such that the following two conditions are fulfilled:

(1) $\pi(xy) = \pi(x)\pi(y)$ and $\pi(1) = I$ (here 1 is the unit element of G and I the identity operator on $\mathfrak{H}$).

(2) The mapping $(x, \psi) \to \pi(x)\psi (x \in G, \psi \in \mathfrak{H})$ is a continuous mapping of $G \times \mathfrak{H}$ into $\mathfrak{H}$.

The object of this note is to announce a few theorems on these representations. No attempt is made to give proofs here. A detailed account with complete proofs will appear elsewhere in another paper.

Let R and C, respectively, be the fields of real and complex numbers. Let $\mathfrak{g}_0$ be the Lie algebra of G and $\mathfrak{g}$ the complexification of $\mathfrak{g}_0$. We denote the universal enveloping[1] algebra of $\mathfrak{g}$ by $\mathfrak{B}$. Let $C_c^\infty(G)$ be the set of all complex-valued functions on G which are indefinitely differentiable everywhere and which vanish outside a compact set. Let π be a representation of G on $\mathfrak{H}$ and V the set of all elements in $\mathfrak{H}$ which can be written as finite linear combinations of elements of the form

$$\int_G f(x)\pi(x)\psi \, dx \qquad (\psi \in \mathfrak{H}, f \in C_c^\infty(G))$$

where dx is the left invariant Haar measure on G. V is called the Gårding subspace[2] of $\mathfrak{H}$ (with respect to π). For every $X \in \mathfrak{g}_0$ we can define a linear transformation $\pi_V(X)$ of V into itself such that

$$\pi_V(x)\psi = \lim_{t \to 0} \frac{1}{t} \{\pi(\exp tX) - I\}\psi \qquad (\psi \in V, t \in R).$$

The mapping $X \to \pi_V(X)$ is a representation of $\mathfrak{g}_0$ and therefore it can be extended uniquely to a representation π_V of $\mathfrak{B}$ on V. Let $\psi \in V$. Consider $\overline{\pi_V(\mathfrak{B})\psi}$ where the bar denotes closure in $\mathfrak{H}$. It turns out that in general $\overline{\pi_V(\mathfrak{B})\psi}$ is not invariant under $\pi(G)$. In order to avoid this unpleasant state of affairs we replace the Gårding subspace V by the space of all well-behaved elements. This is defined as follows. Let U be a subspace of $\mathfrak{H}$ (not necessarily closed). We say that U is well-behaved under π if the following conditions hold.

(1) There exists a representation π_U of $\mathfrak{g}_0$ on U such that

$$\pi_U(X)\psi = \operatorname*{Lim}_{t \to 0} \frac{1}{t} \{\pi(\exp tX) - I\} \psi \qquad (X \in \mathfrak{g}_0, \psi \in U).$$

(2) For any continuous linear function f on $\mathfrak{H}$ and $\psi \in U$ the function

$$f(\pi(x)\psi) \qquad (x \in G)$$

is an analytic function on G. It is clear that if U_1, U_2 are two well-behaved subspaces of $\mathfrak{H}$ then $U_1 + U_2$ is also well-behaved. From this it follows that the union W of all well-behaved subspaces in $\mathfrak{H}$ is itself a well-behaved subspace. An element $\psi \in \mathfrak{H}$ will be called well-behaved if $\psi \in W$ and W is called the space of all well-behaved elements. It is clear that the mapping $X \to \pi_W(X)$ $(X \in \mathfrak{g}_0)$ can be extended uniquely to a representation π_W of $\mathfrak{B}$ on W. The following theorem justifies the notion of well-behaved elements.

THEOREM 1. *Let ψ be a well-behaved element of $\mathfrak{H}$. Then $\overline{\pi_W(\mathfrak{B})\psi}$ is invariant under $\pi(G)$.*

Let G^* be the adjoint group of G and let K^* be a maximal compact subgroup of G^*. Let K be the complete inverse image of K^* in G. Then K is connected though it is not necessarily compact. Let P be the set of all equivalence classes of finite-dimensional irreducible representations of K. Let $\mathfrak{D} \in$ P and $\psi \in \mathfrak{H}$. We say that ψ transforms under $\pi(K)$ according to $\mathfrak{D}$ if the space U spanned by $\pi(u)\psi$ for all $u \in K$ is finite-dimensional and the representation of K induced on U is fully reducible into a direct sum of irreducible components each of which lies in $\mathfrak{D}$. Let $\mathfrak{H}_\mathfrak{D}$ be the set of all elements in $\mathfrak{H}$ which transform under $\pi(K)$ according to $\mathfrak{D}$. Put $W_\mathfrak{D} = W \cap \mathfrak{H}_\mathfrak{D}$. Let Z be the center of G and $\mathfrak{Z}$ the center of $\mathfrak{B}$. Let V be the Gårding subspace of $\mathfrak{H}$ and π_V the representation of $\mathfrak{B}$ on V as defined above. We shall say that π is a quasi-simple representation of G if there exist homomorphisms c and χ of Z and $\mathfrak{Z}$ respectively, into C such that the following two conditions hold:

$$(1) \quad \pi(d) = c(d)I \qquad (d \in Z).$$

$$(2) \quad \pi_V(z)\psi = \chi(z)\psi \qquad (z \in \mathfrak{Z}, \psi \in V).$$

In case condition (2) is satisfied we say that π has the character χ.

THEOREM 2. *Let π be a quasi-simple representation of G on $\mathfrak{H}$. Then $\sum_{\mathfrak{D} \in P} W_{\mathfrak{D}}$ is dense in $\mathfrak{H}$ and $\mathfrak{H}_{\mathfrak{T}} = \overline{W_{\mathfrak{D}}}$ $(\mathfrak{B} \in P)$.*

Here $\sum_{\mathfrak{D} \in P} W_{\mathfrak{D}}$ is the space consisting of all finite linear combinations of elements in $\bigcup_{\mathfrak{D} \in P} W_{\mathfrak{D}}$.

THEOREM 3. *Let π be quasi-simple and let $\psi \in \sum_{\mathfrak{D} \in P} W_{\mathfrak{D}}$. Put $U = \overline{\pi_W(\mathfrak{B})\psi}$ and $U_{\mathfrak{T}} = U \cap \mathfrak{H}_{\mathfrak{D}}(\mathfrak{D} \in P)$. Then $\pi_W(\mathfrak{B})\psi = \sum_{\mathfrak{D} \in P} U_{\mathfrak{T}}$ and dim $U_{\mathfrak{T}} < \infty$ for every $\mathfrak{D} \in P$.*

Let U, V be two subspaces of $\mathfrak{H}$. We write $U > V$ or $V < U$ if $U \supset V$ and $U \neq V$. Let U and V be two closed subspaces of $\mathfrak{H}$ invariant under $\pi(G)$. We shall say that V is maximal in U if $U > V$ and there exists no closed invariant subspace U_1 such that $U > U_1 > V$.

THEOREM 4. *Let π be a quasi-simple representation of G on a Banach space $\mathfrak{H} \neq \{0\}$. Then there exist two closed invariant subspaces U and V in $\mathfrak{H}$ such that V is maximal in U.*

Now we shall consider the special case when $\mathfrak{H}$ is a Hilbert space and π is a unitary representation. From Theorem 4 we deduce the following result.

THEOREM 5. *Let π be a quasi-simple unitary representation of G on a Hilbert space $\mathfrak{H} \neq \{0\}$. Then there exists a closed invariant subspace U of $\mathfrak{H}$ such that $\{0\}$ is maximal in U (i.e., $U > \{0\}$ and U is irreducible under $\pi(G)$).*

The above theorem has the following significance in relation to the theory of factors of Murray and von Neumann.[3] Let π be a unitary representation of G on a Hilbert space $\mathfrak{H}$. Let $\mathfrak{A}$ be the smallest weakly closed algebra of bounded operators on $\mathfrak{H}$ containing $\pi(G)$. It is known[4] that if $\mathfrak{A}$ is a factor, i.e., if the center of $\mathfrak{A}$ consists of scalar multiples of I, then π is quasi-simple. Therefore in this case Theorem 5 is applicable and so it follows from the results of Murray and von Neumann[3] that $\mathfrak{A}$ must be a factor of type I_n or I_∞. This shows that factors of Type II and Type III cannot arise from a unitary representation of a semisimple Lie group on a Hilbert space. For the consequences of this result we refer the reader to the above-quoted paper of Mautner.

Let $\mathfrak{D} \in P$. We say that $\mathfrak{D}$ occurs in π if $\mathfrak{H}_{\mathfrak{D}} \neq \{0\}$. It is known that all unitary irreducible representations of G are quasi-simple.

THEOREM 6. *Let χ be a homomorphism of $\mathfrak{Z}$ into C such that $\chi(1) = 1$ and let $\mathfrak{D} \in P$. Let $\mathfrak{E}(\chi, \mathfrak{D})$ denote the set of all representations π of G which have the following properties:*

(1) *π is an irreducible unitary representation of G on some Hilbert space.*

(2) *$\mathfrak{D}$ occurs in π.*

(3) π *has the character* χ.

Then $\mathfrak{E}(\chi, \mathfrak{D})$ *contains only a finite number of inequivalent representations*

Let $d(\mathfrak{D})$ denote the degree of any representation in $\mathfrak{D}$. Suppose $d(\mathfrak{D}) = 1$, $\pi \in \mathfrak{E}(\chi, \mathfrak{D})$ and $\mathfrak{H}$ is the representation space of π. Then it can be shown that $\dim \mathfrak{H}_{\mathfrak{T}} = 1$ and the maximum number of inequivalent representations in $\mathfrak{E}(\chi, \mathfrak{D})$ is not greater than the order of the Weyl group of $\mathfrak{g}$ with respect to some Cartan subalgebra. Moreover if G is a complex semi-simple Lie group all representations in $\mathfrak{E}(\chi, \mathfrak{D})$ are equivalent (provided $d(\mathfrak{D}) = 1$).

Let π be an irreducible unitary representation of G on a Hilbert space $\mathfrak{H}$. For any $\mathfrak{D} \in \mathrm{P}$ let $E_{\mathfrak{D}}$ denote the orthogonal projection of $\mathfrak{H}$ on $\mathfrak{D}_{\mathfrak{T}}$. Since π is irreducible and unitary it is quasi-simple and therefore it follows from Theorem 4 that $\dim \mathfrak{H}_{\mathfrak{T}} < \infty$. Hence the function $\varphi_{\mathfrak{D}}^{\pi}(x) = sp(E_{\mathfrak{D}}\, \pi(x) E_{\mathfrak{D}})(x \in G)$ is well defined. It is an analytic function on G and $\varphi_{\mathfrak{D}}^{\pi}(u\, x\, u^{-1}) = \varphi_{\mathfrak{D}}^{\pi}(x)$ $(u \in K)$.

THEOREM 7. *Let* π_1 *and* π_2 *be irreducible unitary representations of* G *on two Hilbert spaces. Let* P_{π_1} *be the set of all elements in* P *which occur in* π_1. *Suppose that for some* $\mathfrak{D} \in \mathrm{P}_{\pi_1}$ *and* $c \in C$,

$$\varphi_{\mathfrak{D}}^{\pi_1} = c\varphi_{\mathfrak{D}}^{\pi_2}.$$

Then π_1 *and* π_2 *are equivalent. Conversely if* π_1 *and* π_2 *are equivalent*

$$\varphi_{\mathfrak{D}}^{\pi_1} = \varphi_{\mathfrak{D}}^{\pi_2}$$

for all $\mathfrak{D} \in \mathrm{P}$.

In case $d(\mathfrak{D}) = 1$ the function $\varphi_{\mathfrak{D}}^{\pi}$ is the same as the spherical function introduced by Gelfand and Naimark.[5] Now suppose G is a complex semi-simple Lie group and $d(\mathfrak{D}) = 1$. Then we have seen above that the representation π is completely determined within equivalence by its character χ. Actually in this case an explicit formula for $\varphi_{\mathfrak{D}}^{\pi}$ in terms of χ can be obtained quite easily. This formula is very similar to the one given by Gelfand and Naimark[5] for representations of the principal series.

[1] See, for example, Harish-Chandra, *Ann. Math.*, **50**, 900–915 (1949).

[2] Garding, L., PROC. NATL. ACAD. SCI., **33**, 331–332 (1947).

[3] Murray, F. J., and von Neumann, J., *Ann. Math.*, **37**, 116–229 (1936).

[4] See Mautner, F. I., *Ann. Math.*, **52**, 528–556 (1950).

[5] Gelfand, I. M., and Naimark, M. A., *Doklady Akad. Nauk SSR (N. S.)*, **63**, 225–228 (1948).

A NON-REFLEXIVE BANACH SPACE ISOMETRIC WITH ITS SECOND CONJUGATE SPACE

BY ROBERT C. JAMES

UNIVERSITY OF CALIFORNIA

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A Banach space B is isometric with a subspace of its second conjugate space B^{**} under the "natural mapping" for which the element of B^{**} which corresponds to the element x_0 of B is the linear functional F_{x_0} defined by $F_{x_0}(f) = f(x_0)$ for each f of B^*. If every F of B^{**} is of this form, then B is said to be reflexive and B is isometric with B^{**} under this natural mapping. The purpose of this note is to show that B can be isometric with B^{**} without being reflexive. The example given to show this is a space isomorphic with a Banach space known to not be reflexive, but to be isomorphic with its second conjugate space.[1]

A sequence $\{x^n\}$ of elements of a Banach space B is said to be a *basis* for B if for each x of B there is a unique sequence of numbers $\{a_n\}$ such that $x = \sum_1^\infty a_i x^i$ in the sense that $\lim_{n\to\infty} \|x - \sum_1^n a_i x^i\| = 0$. A fundamental sequence $\{x^n\}$ is a basis if and only if there is a positive number ϵ such that $\|\sum_1^{n+p} a_i x^i\| \geq \epsilon \|\sum_1^n a_i x^i\|$ for any positive integers n and p and numbers $\{a_i\}$.[2] If $\epsilon = 1$, the basis will be called an *orthogonal basis*. But for any basis $\{x^n\}$, $|||x||| = \sup_n \|\sum_1^n a_i x^i\|$ for $x = \sum_1^\infty a_i x^i$ defines a new norm $|||\ \ |||$ which is equivalent to $\|\ \ \|$ and for which $\{x^n\}$ is an orthogonal basis.[3] Hence if B has a basis $\{x^n\}$ for which $\lim_{n\to\infty} \|f\|_n = 0$ for each f of B^*, where $\|f\|_n$ is the norm of f on $x^{n+1} \oplus x^{n+2} \oplus \ldots$, then the following theorem describes B^{**} completely if the basis is orthogonal and describes B^{**} to within an isomorphism if the basis is not an orthogonal basis.

THEOREM. *Let B be a Banach space with an orthogonal basis $\{z^n\}$ for which $\lim_{n\to\infty} \|f\|_n = 0$ for each f of B^*, where $\|f\|_n$ is the norm of f on $z^{n+1} \oplus z^{n+2} \oplus \ldots$. Then $\{g^n\}$ is a basis for B^* if $g^n(z^m) = \delta_m^n$ for each n and m. If $F \in B^{**}$, then $\|F\| = \lim_{n\to\infty} \|\sum_1^n F_i z^i\|$, where $F_i = F(g^i)$. If the sequence $\{F_n\}$ is such that $\lim_{n\to\infty} \|\sum_1^n F_i z^i\| < +\infty$, then $F \in B^{**}$ if one defines $F(f) = \sum_1^\infty F_i f_i$ for each $f = \sum_1^\infty f_i g^i$ of B^*.*

Proof: It has been previously known that $\{g^n\}$ is a basis for B^*.[4] It

follows from this that $F(f) = \sum_1^\infty F_i f_i$ for each F of B^{**} and each $f = \sum_1^\infty f_i g^i$ of B^*, where $F_i = F(g^i)$. But, for each $f = \sum_1^\infty f_i g^i$, $|\sum_1^n F_i f_i| = |f(\sum_1^n F_i z^i)| \leq \|f\| \|\sum_1^n F_i z^i\|$. Thus $|\sum_1^\infty F_i f_i| \leq \|f\| (\lim_{n\to\infty} \|\sum_1^n F_i z^i\|)$, and $\|F\| \leq \lim_{n\to\infty} \|\sum_1^n F_i z^i\|$. For a fixed n, let $u^n = \sum_1^n F_i z^i$. Define a linear functional h by $h(z^i) = 0$ for $i > n$ and $h(u^n) = \|u^n\|$. Then $|h(au^n + \sum_{n+1}^\infty a_i z^i)| = \|au^n\| \leq \|au^n + \sum_{n+1}^\infty a_i z^i\|$. Thus $\|h\| = 1$ on $u^n \oplus z^{n+1} \oplus z^{n+2} \oplus \ldots$. Extend h to all of B so that $\|h\| = 1$ on B. Then, for this h, $h = \sum_1^\infty h_i g^i$ with $h_i = 0$ for $i > n$, so that $|\sum_1^\infty F_i h_i| = |\sum_1^n F_i h_i| = |h(u^n)| = \|u^n\| \leq \|F\|$. Since this can be done for each n, it follows that $\|F\| \geq \|\sum_1^n F_i z^i\|$ for each n and $\|F\| \geq \lim_{n\to\infty} \|\sum_1^n F_i z^i\|$. It has thus been shown that $\lim_{n\to\infty} \|\sum_1^n F_i z^i\| = \|F\|$ for each element $F = \{F_n\}$ of B^{**}. Now suppose that $\{F_n\}$ is a sequence such that $\lim_{n\to\infty} \|\sum_1^n F_i z^i\| = M < +\infty$. Then $\|\sum_n^{n+p} F_i z^i\| \leq 2M$. Thus for any fixed $f \in B^*$, $|\sum_n^{n+p} F_i f_i| = |f(\sum_n^{n+p} F_i z^i)| \leq \|f\|_n (2M)$, so that it follows from $\lim_{n\to\infty} \|f\|_n = 0$ that $\sum_1^\infty F_i f_i$ is convergent. Thus $F(f) = \sum_1^\infty F_i f_i$ is defined for each $f \in B^*$ and $\|F\| = \lim_{n\to\infty} \|\sum_1^n F_i z^i\|$.

Example: *For* $x = (x_1, x_2, x_3, \ldots)$, *let*

$$\|x\| = l.\ u.\ b. \left[\sum_{i=1}^n (x_{p_i} - x_{p_{i+1}})^2 + (x_{p_{n+1}} - x_{p_1})^2\right]^{1/2}, \qquad (1)$$

where the l. u. b. is over all positive integers n and all finite increasing sequences of at least two positive integers $p_1, p_2, \ldots, p_{n+1}$. *Let B be the Banach space of all x for which* $\|x\|$ *is finite and* $\lim_{n\to\infty} x_n = 0$. *Then B is isometric with* B^{**}, *but is isometric under the natural mapping with a closed maximal linear subspace of* B^{**}.

Proof: For $x = (x_1, x_2, \ldots)$, let

$$|||x||| = \text{l. u. b.} \left[\sum_{i=1}^n (x_{p_{2i-1}} - x_{p_{2i}})^2 + (x_{p_{2n+1}})^2\right]^{1/2}, \qquad (2)$$

where the l. u. b. is over all positive integers n and finite increasing se-

quences of positive integers $p_1, p_2, \ldots, p_{2n+1}$. It follows from $\lim_{n\to\infty} x_n = 0$ and $\|x\| \geq |x_n - x_p|$ that $\|x\| \geq |x_p|$ for each p. Clearly $|||x||| \geq |x_p|$ for each p. But by grouping alternating terms of (1) and isolating x_{p_1}, one gets $\|x\| \leq$ l. u. b. $\{|x_{p_1}| + [(x_{p_{n+1}})^2 + (x_{p_n-1} - x_{p_n})^2 + (x_{p_n-3} - x_{p_n-2})^2 + \ldots]^{1/2} + [(x_{p_n} - x_{p_{n+1}})^2 + (x_{p_n-2} - x_{p_n-1})^2 + \ldots]^{1/2}\} \leq 3|||x|||$. But extra terms can be introduced in (2) to give a sum of type (1), except for replacing $(x_{p_{2n+1}})$ by $(x_{p_{2n+1}} - x_{p_1})$. Thus $|||x||| \leq 2\|x\|$. Since $\frac{1}{2}|||x||| \leq \|x\| \leq 3|||x|||$, these two norms are equivalent. But the Banach space of all $x = (x_1, x_2, \ldots)$ for which $\lim_{n\to\infty} x_n = 0$ and $|||x|||$ is finite is known to not be reflexive, but to be isometric under the natural mapping with a closed maximal linear subspace of its second conjugate space.[1] Hence this is also true of the space B.

Let $z^n = (0, 0, \ldots, 0, 1, 0, \ldots)$ be the element of B whose components are all zero except for the nth component, which is 1. Then $z^1 \oplus z^2 \oplus \ldots = B$, so that $\{z^n\}$ is an orthogonal basis for B if $\|\sum_1^n a_i z^i + \sum_{n+1}^{n+p} b_i z^i\| \geq \|\sum_1^n a_i z^i\|$ for all numbers $\{a_i\}$ and $\{b_i\}$ and positive integers n and p. Since $\sum_1^n a_i z^i$ has only a finite number of non-zero components, a sequence $p_1, p_2, \ldots, p_{k+1}$ can be chosen so that

$$\|\sum_1^n a_i z^i\| = \left[\sum_{i=1}^{k} (a_{p_i} - a_{p_{i+1}})^2 + (a_{p_{k+1}} - a_{p_1})^2\right]^{1/2}, \qquad (3)$$

where $a_r = 0$ if $r > n$. If $p_{k+1} \leq n$, then it is immediate from (1) and (3) that $\|\sum_1^n a_i z^i + \sum_{n+1}^{n+p} b_i z^i\| \geq \|\sum_1^n a_i z^i\|$. If $p_{k+1} > n$, then each p_i with $p_i > n$ can be replaced by some $p_i > n + p$ without changing the value of (3), since $a_r = 0$ if $r > n$. But it will then again follow from (1) and (3) that $\|\sum_1^n a_i z^i + \sum_{n+1}^{n+p} b_i z^i\| \geq \|\sum_1^n a_i z^i\|$. For B with the norm $|||\ \ |||$, and hence also for B with the norm $\|\ \ \|$, it is known that $\lim_{n\to\infty} \|f\|_n = 0$, where $\|f\|_n$ is the norm of f on $z^{n+1} \oplus z^{n+2} \oplus \ldots$.[1] Hence by Theorem 1 above, B^{**} is the space of all $F = (F_1, F_2, \ldots)$ for which $\|F\| = \lim_{n\to\infty} \|\sum_1^n F_i z^i\|$ is finite. Thus for F to belong to B^{**}, it is necessary that $\lim_{n\to\infty} F_n$ exist. Consider the correspondence:

$$x = (x_1, x_2, \ldots) \longleftrightarrow (x_2 - x_1, x_3 - x_1, \ldots, x_n - x_1, \ldots) =$$
$$(F_1, F_2, \ldots) = F_x.$$

To show that $\|x\| = \|F_x\|$, first consider a sum $\left[\sum_{i=1}^{n}(x_{p_i} - x_{p_{i+1}})^2 + (x_{p_{n+1}} - x_{p_1})^2\right]$. If $p_1 \geq 2$, this is equal to $\left[\sum_{i=1}^{n}(F_{p_i-1} - F_{p_{i+1}-1})^2 + (F_{p_{n+1}-1} - F_{p_1-1})^2\right]$. If $p_1 = 1$, it is equal to $\left[\sum_{i=2}^{n}(F_{p_i-1} - F_{p_{i+1}-1})^2 + (F_{p_{n+1}-1} - F_N)^2 + (F_N - F_{p_2-1})^2\right]$, if $N > p_{n+1} - 1$ and F_N is replaced by zero. Since $\|\sum_1^n F_i z^i\|$ is a monotonically increasing function of n, it follows that $\|x\| \leq \|F_x\|$, where $\|F_x\| = \lim_{n\to\infty}\|\sum_1^n F_i z^i\|$. Now consider a sum $\left[\sum_{i=1}^{n}(F_{p_i} - F_{p_{i+1}})^2 + (F_{p_{n+1}} - F_{p_1})^2\right]$, formed for the element $\sum_1^m F_i z^i$, where F_p is to be replaced by 0 if $p > m$. If $p_{n+1} \leq m$, then this sum is equal to $\left[\sum_{i=1}^{n}(x_{p_i+1} - x_{p_{i+1}+1})^2 + (x_{p_{n+1}+1} - x_{p_1+1})^2\right]$. Now suppose that $p_{k+1} > m$, but $p_i \leq m$ if $i \leq k$. Then the sum becomes $\left[\sum_{i=1}^{k-1}(F_{p_i} - F_{p_{i+1}})^2 + (F_{p_k})^2 + (F_{p_1})^2\right] = \left[\sum_{i=1}^{k-1}(x_{p_i+1} - x_{p_{i+1}+1})^2 + (x_{p_k+1} - x_1)^2 + (x_1 - x_{p_1+1})^2\right]$. Thus $\|x\| \geq \|\sum_1^n F_i z^i\|$ for each n. Hence $\|x\| = \|F_x\|$ and $x \longleftrightarrow F_x$ is an isometry with domain equal to B. But if $F = (F_1, F_2, \ldots)$ is an element of B^{**}, and $\lim_{n\to\infty} F_n = L$, then $x_F = (-L, F_1 - L, F_2 - L, \ldots)$ is, by the above, an element of B for which $\|x_F\| = \|F\|$ and $x_F \longleftrightarrow F$. Thus the range of the isometry is B^{**}.

[1] James, R. C., "Bases and Reflexivity of Banach Spaces," *Ann. Math.*, **52**, 518–527 (1950).

[2] Grinblum, M. M., "Certain théorèmes sur la base dans un espace du type (B)," *C. R. (Doklady) Acad. Sci. URSS* (N. S.), **31**, 428–432 (1941).

[3] Banach, S., *Théorie des Opérations Linéaires*, Warsaw, 1932, p. 111.

[4] James, *loc. cit.*, Theorem 3.

PROBABILISTIC THEORIES OF RELATIONS

BY KARL MENGER

ILLINOIS INSTITUTE OF TECHNOLOGY

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Poincaré repeatedly emphasized that only in the mathematical continuum the equalities $A = B$ and $B = C$ imply the equality $A = C$. In the observable physical continuum, "equal" means "indistinguishable," and $A = B$ and $B = C$ by no means imply $A = C$. "The raw result of experience may be expressed by the relation

$$A = B, B = C, A < C$$

which may be regarded as the formula for the physical continuum." According to Poincaré, physical equality is a non-transitive relation.

A closer examination of the physical continuum suggests that in describing our observations we should sacrifice more than the transitivity of equality. We should give up the assumption that equality is a relation. For this assumption implies that for every two elements, A and B, the question as to whether or not A and B can be distinguished, is inalterably settled. But simple experiments show that, for instance, the simultaneous irritation of the same two spots, A and B, on the skin sometimes produces one sensation, sometimes two. Only by reliance on the majority of the impressions, by processes of averaging and the formation of means, equality relations have been artificially created.

We obtain a more realistic theoretical description of the equality of two elements *by associating with A and B a number, namely, the probability of finding A and B indistinguishable.* In applications, this number would be represented by the relative frequency of the cases in which A and B are not distinguished.

In principle, this idea solves Poincaré's paradox. For if it is only very likely that A and B are equal, and very likely that B and C are equal, why should it not be less likely that A and C are equal? In fact, why should the equality of A and C not be less likely than the inequality of A and C?

If $E(a, b)$ denotes the probability that a and b be equal, the following postulates seem to be rather natural:

(1) $E(a, a) = 1$ for every a;
(2) $E(a, b) = E(b, a)$ for every a and b;
(3) $E(a, b) \cdot E(b, c) \leq E(a, c)$ for every a, b, c.

(1) and (2) correspond to the reflexivity and symmetry of the equality relation, (3) expresses a minimum of transitivity.

If we call a and b *certainly-equal* provided that $E(a, b) = 1$, we obtain

an equality relation. All the elements which are certainly-equal to a may be united in an "equality-set," A. Any two such sets are disjoint unless they are identical. We may define $E(A, B)$ as the probability that any element of A and any element of B be equal. For this number is independent of the particular choice of the two elements.

If we set $-\log E(A, B) = d(A, B)$, then we readily conclude:

$(1_a')$ $d(A, A) = 0$; $(1')$ $d(A, B) \geq 0$; $(1_c')$ $d(A, B) \neq 0$ if $A \neq B$;
$(2')$ $d(A, B) = d(B, A)$;
$(3')$ $d(A, B) + d(B, C) \geq d(A, C)$.

These are Fréchet's postulates for the distance in a metric space. In particular $(3')$ is the triangle inequality. Conversely, if disjoint sets $A, B, \ldots$ form a metric space with the distance $d(A, B)$ and we set $E(A, B) = E(a, b) = e^{-d(A, B)}$ for each element a of A, and b of B, then $E(a, b)$ satisfies the postulates (1), (2), (3), of a probability of equality. *The systems of probabilities of equality in a set, S, are thus identical with the systems of negative antilogarithms of the distance for the various possible metrizations of S.*

If S is a straight line, $E(a, b) = e^{-|b - a|}$ is the probability of equality corresponding to the euclidean metrization of S. Another example is $E(a, b) = e^{-\sqrt{|b-a|}}$ while the smooth function $e^{-(b - a)^3}$ is ruled out by the fact that the corresponding distance $d(a, b) = (b - a)^2$ fails to satisfy the triangle inequality.

Since $d(a, b) < \infty$, for every two points of a metric space, it follows that $E(a, b) > 0$. We may find it more desirable to assume that every element b differing from a by more than a certain exterior threshold be certainly distinguishable from a. But *then we have to give up even the minimum of transitivity expressed in Postulate (3).*

In the same way as we have just treated equality we may treat the order relation "a precedes b." If we introduce the probability $P(a, b)$ of a preceding b, the following postulates are natural.

Monotony. If $a_1 \leq a_2 \leq b_2 \leq b_1$, then $P(a_2, b_2) \leq P(a_1, b_1)$.

Boundary Conditions. $\lim_{y \to \infty} P(a, y) = \lim_{x \to -\infty} P(x, b) = 1$.

The following assumptions reflect properties of the order relation:

Asymmetry. $P(a, b) + P(b, a) = 1$.

Transitivity. $P(a, b) \cdot P(b, c) \leq P(a, c)$.

An example of a function satisfying the above postulates is

$$P(a, b) = \begin{cases} k^{(b - a)/2} & \text{if} \quad b \leq a \\ 1 - k^{(a - b)/2} & \text{if} \quad a \leq b \end{cases}$$

for any $k > 1$. Besides, this function satisfies the following condition of

Homogeneity. $P(a, b) = P(a', b')$ if $b' - a' = b - a$.

Moreover, the example illustrates the following general fact. If we

wish to satisfy the transitivity postulate we must admit that every number, even a very large one, has a positive probability of preceding 0.

It is clear that probabilistic theories of dyadic, triadic, and higher relations can be developed which are based on functions of 2, 3, and more variables. *This procedure seems to be a natural extension of the idea of replacing qualities by quantities* (monadic relations by functions of one variable).

But the probabilistic point of view far transcends the domain of relations. It is applicable to functions. Instead of a real function of one, two or n variables, we may consider *the association of a distribution function with every element, pair or n-tuple of elements of a set.* Of particular geometric importance is the probabilistic generalization of the concept of a metric space.[1]

[1] An attempt in this direction was made by the author in the note "Statistical Metrics," PROC. NATL. ACAD. SCI., **28**, 535–537 (1942).

HARMONIC FORMS AND HEAT CONDUCTION

I: CLOSED RIEMANNIAN MANIFOLDS

BY A. N. MILGRAM AND P. C. ROSENBLOOM

DEPARTMENT OF MATHEMATICS, SYRACUSE UNIVERSITY

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This paper is concerned with properties of exterior differential forms defined on a Riemannian manifold V_n of class C^r, $r \geq 5$. The existence and uniqueness of harmonic forms having prescribed periods has for some time been known.[1–5] In this note an analog to the heat equation for exterior differential forms is studied and the theory of harmonic forms follows as a limiting case analogous to the steady state in heat flow. We restrict our considerations in the present note to closed orientable manifolds reserving for future publication the adaptation of our method to open manifolds.

In the following we shall adopt the notation used in reference 3 with the following minor modification: $\delta\alpha = (-1)^{np+n}(*d*\alpha)$, where p is the degree of α. Then $\Delta\alpha = (d\delta + \delta d)\alpha$ is merely the negative of the operator used in reference 5, and the analog of Green's formula has the following form: $(d\alpha, \beta) = -(\alpha, \delta\beta)$. The symbol $\|\alpha\|$ will denote the norm, where $\|\alpha\| = + \sqrt{(\alpha, \alpha)}$.

If α is a form whose coefficients depend on a parameter t, by $\dfrac{\partial\alpha}{\partial t}$ we mean the form obtained by replacing each coefficient of α by its partial

derivative with respect to t. Our principal concern will be with the existence and behavior of solutions of the equations

$$L\,\alpha = \Delta\alpha - \frac{\partial\alpha}{\partial t} = 0 \tag{1}$$

and

$$L\,\alpha = \beta \tag{1'}$$

where β is a given form.

Let $L_1 = \Delta + \dfrac{\partial}{\partial t}$, the operator adjoint to L. Using $(\alpha, \Delta\beta) - (\beta, \Delta\alpha) = 0$, we remark that

$$(L_1\beta, \alpha) - (\beta, L\,\alpha) = \frac{\partial}{\partial t}(\alpha, \beta). \tag{2}$$

We now define a parametrix adapted to the operator L. Let r denote a distance function (cf. 3); let

$$\beta_p = \sum A_{i_1\cdots i_p;\ j_1\cdots j_p} dx^{i_1}\cdots dx^{i_p} dy^{j_1}\cdots dy^{j_p}$$

where $A_{i_1\cdots i_p,\ j_1\cdots j_p} = \det |A_{i_\rho j_\sigma}|$, $A_{ij} = \dfrac{\partial^2 A}{\partial x^i \partial y^j}$ and $A = -r^2/2$; and let

$$w_p(x, y, t) = Kt^{-n/2} e^{A/2t}\beta_p,\ K = 2^{-n}\pi^{-n/2}. \tag{3}$$

On replacing β in (2) by $w_p(x, y, t-\tau)$ and regarding x and t as parameters in the differentiations and integrations, we obtain for $0 < h < t$

$$\int_0^{t-h} \{(L_1 w_p(x, y, t-\tau), \alpha(y, \tau)) - (w_p(x, y, t-\tau), L\,\alpha(y, \tau))\}\, d\tau = (\alpha(y, t-h), w_p(x, y, h)) - (\alpha(y, 0), w_p(x, y, t)). \tag{4}$$

We now let h approach zero.

At this point we need the following lemmas:

LEMMA 1. *The norm of $L_1 w_p(x, y, t)$ is bounded by a constant independent of x and t.*

LEMMA 2. *If α is continuous, then* $\lim_{h\to 0} (\alpha(y, t), w_p(x, y, h)) = \alpha(x, t)$.

The proofs of these lemmas involve the use of osculating Euclidean coordinate systems as in reference 3 combined with the standard techniques given in references 6 and 7. Actually by these simple computations it is shown that the kernel $L_1 w_p(x, y, t-\tau)$ yields a bounded operator in all the classical norms.

Thus for twice differentiable forms α we arrive at

$$\int_0^t \{(L_1 w_p(x, y, t-\tau), \alpha(y, \tau)) - (w_p(x, y, t-\tau), L\,\alpha(y, \tau))\}\, d\tau = \alpha(x, t) - (\alpha(y, 0), w_p(x, y, t)). \tag{5}$$

Hence if α is a solution of (1'), then α satisfies the Volterra integral equation

$$\alpha = \gamma + \int_0^t (L_1 w_p, \alpha) d\tau, \tag{6}$$

where

$$\gamma = (\alpha(y, 0), w_p(x, y, t)) - \int_0^t (w_p(x, y, t - \tau), \beta(y, \tau)) d\tau \tag{7}$$

THEOREM 1. *If $\alpha(y, 0)$ and β are given forms of class C, and γ is defined by (7), then the integral equation (6) has a unique solution, which is of class C^r for $t > 0$ and continuous for $t \geq 0$. This solution of (6) is also the unique solution of (1') taking on the prescribed initial value $\alpha_0 = \alpha(x, 0)$ for $t = 0$.*

The existence and uniqueness of the solution of (6) follows trivially by the method of successive approximations, as in the standard theory of Volterra's integral equations. The regularity properties of the solution of (6) may be proved directly from the integral equation or by examination of the resolvent kernel, which is, of course, the fundamental solution of the heat equation. This theorem plays a central role, and dispenses with the much more complicated machinery based on the Fredholm theory, which was used by previous authors.[1-5]

Let $T_t\alpha$ denote the solution of (1) having initial value $\alpha(x)$ at $t = 0$. Then since (1) is invariant under translations with respect to t and by the uniqueness of the solution of (1) it follows that $\{T_t\}$ is a semigroup; i.e., $T_t T_h = T_{t+h}$ (cf. 8). Moreover if $g(t, \tau) = (T_t\alpha_1, T_\tau\alpha_2)$, then

$$\frac{\partial}{\partial t} g(t, \tau) = \left(\frac{\partial}{\partial t} T_t\alpha_1, T_\tau\alpha_2\right) = (\Delta T_t\alpha_1, T_\tau\alpha_2) = (T_t\alpha_1, \Delta T_\tau\alpha_2) =$$

$$\left(T_t\alpha_1, \frac{\partial}{\partial \tau} T_\tau\alpha_2\right) = \frac{\partial}{\partial \tau} g(t, \tau),$$

so that $g(t, \tau)$ is a function of $t + \tau$. Observe that this implies T_t is a symmetric operator, for setting $\tau = 0$ gives $(T_t\alpha_1, \alpha_2) = (\alpha_1, T_t\alpha_2)$. Also

$$\|T_{t+2h}\alpha - T_t\alpha\|^2 = \|T_{t+2h}\alpha\|^2 + \|T_t\alpha\|^2 - 2(T_{t+2h}\alpha, T_t\alpha) =$$

$$\|T_{t+2h}\alpha\|^2 + \|T_t\alpha\|^2 - 2\|T_{t+h}\alpha\|^2.$$

But $\|T_t\alpha\|$ is a non-negative, non-increasing function of t, for

$$\frac{d}{dt}(T_t\alpha, T_t\alpha) = 2\left(\frac{\partial}{\partial t} T_t\alpha, T_t\alpha\right) = 2(\Delta T_t\alpha, T_t\alpha) =$$

$$-2(dT_t\alpha, dT_t\alpha) - 2(\delta T_t\alpha, \delta T_t\alpha) \leq 0.$$

Hence not only does the limit of $\|T_t\alpha\|$ exist as $t \to \infty$ but also $T_t\alpha$ converges in the mean to a form $H\alpha$. From Lemma 1 and the relation $T_{t+h}\alpha = T_t T_h\alpha = T_h T_t\alpha$ we see that convergence in the mean implies uniform convergence and $H\alpha = T_t H\alpha$ which by Theorem 1 shows that $H\alpha$ is of class C^r, and that $\Delta H\alpha = \dfrac{\partial}{\partial t} H\alpha = 0$, so that $H\alpha$ is a harmonic

form.

Noting that $\Delta d = d\Delta$, $d\dfrac{\partial}{\partial t} = \dfrac{\partial}{\partial t} d$ it follows that if $\alpha(x, t)$ is a solution of (1), then $d\alpha$ is also. Hence if $\alpha(x, 0)$ is closed $d\alpha$ is a solution of (1) having initial value 0, and by the uniqueness theorem we have $d\alpha \equiv 0$ for all t. We conclude that if a form is closed initially, it remains closed for all time.

If α is a solution of (1), then

$$\frac{\partial}{\partial t}\int_C \alpha = \int_C \Delta\alpha = \int_C \delta d\alpha + \int_{\partial C} \delta\alpha$$

where C is any p chain in V_n and ∂C denotes the boundary of C. If α is closed and C is a cycle, the last two terms vanish so that $\int_C \alpha$ is independent of t, and therefore also $\int_C \alpha = \int_C H\alpha$.

THEOREM 2. *If α is a solution of (1) which is closed for $t = 0$, then it is closed for all time. Its periods remain constant, and as $t \to \infty$ α converges uniformly to a harmonic form $H\alpha$ having the same periods.*

This shows that Hodge's main theorem on the existence of harmonic forms with prescribed periods follows immediately from de Rham's theorem[9] on the existence of closed forms with prescribed periods. (We recently learned in conversations with B. Eckmann and W. Hurewicz that they have found independently very simple proofs of the latter theorem, so that combining their work with the present note provides a new approach to the whole theory.)

We now consider the variational problem of finding for a fixed $t > 0$, the maximum of $\|T_t\alpha\|$ under the conditions $\|\alpha\| = 1$, $H\alpha = 0$. From the integral representation of T_t in terms of the resolvent kernel of (6), we see immediately that T_t is a completely continuous operator, so that this maximum is actually attained. If $\alpha(t)$ is an extremal form for this problem and $\lambda(t)$ is the corresponding maximum, then a standard argument shows that $T_{2t}\alpha = \lambda(t)^2\alpha$. Straightforward considerations now show that $\lambda(t) = e^{-\lambda_1 t}$ where λ_1 is a certain positive constant.

An elementary calculation shows that the solution of (1′) with the initial value α_0 is given by the formula

$$\alpha = T_t\alpha_0 - \int_0^t T_{t-\tau}\beta d\tau. \tag{8}$$

If β is independent of t, then this takes the simpler form

$$\alpha = T_t\alpha_0 - \int_0^t T_\tau\beta d\tau. \tag{8′}$$

But $$\|T_\tau\beta - H\beta\| \leq e^{-\lambda_1\tau},$$

which is integrable over $[0, \infty]$, so that

$$\lim_{t\to\infty} (\alpha + tH\beta) = \gamma \tag{9}$$

exists uniformly, and it is readily seen that

$$\Delta\gamma = \beta - H\beta. \tag{10}$$

Equation (10) implies the fundamental decomposition theorem of Hodge.

The operators $U_t = e^{\lambda_1 t}(T_t - H)$ form a semigroup with properties similar to those of $\{T_t\}$, and these converge, as $t \to \infty$, to a projection H_1 onto the solutions of $\Delta\alpha = -\lambda_1\alpha$. By repeating this procedure we obtain the spectral resolution of T_t in the form

$$T_t = H + \sum_{k=1}^{\infty} e^{-\lambda_k t} H_k,$$

where $\{\lambda_k\}$ are the eigenvalues of the equation $\Delta\alpha = -\lambda\alpha$. The completeness of the eigenfunctions is quite easy to prove directly.

We note that another generalization of the Laplace operator was introduced in reference 10, and also that other treatments of parabolic equations occur in references 11 and 12.

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NATIONAL ACADEMY OF SCIENCES CONFERENCE ON COMPLEMENT

BY MICHAEL HEIDELBERGER

COLLEGE OF PHYSICIANS & SURGEONS, COLUMBIA UNIVERSITY

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In the spring of 1950 the writer undertook to organize a conference, to be sponsored by the National Academy of Sciences, on some phase of immunology. As a result of discussions on suitable topics it was decided to assemble a group, with a wide diversity of interests, in order to throw light from all sides on the subject of *complement*.

The reasons for the choice of topic were as follows:

Following the introduction of rigorous, quantitative analytical micromethods for the estimation, in weight units, of antigens and antibodies the part played by these substances and their modes of interaction in immunity to infectious disease became greatly clarified.[1] Much progress has resulted from the application of similar methods to the study of complement,[2] which, next to antigens and antibodies, is the most important auxiliary substance concerned with immunity. However, the complexity of complement and its associated phenomena frequently made difficult the interpretation of analytical data.

Often the mere combination of antibodies with the infectious agent is insufficient to set in motion the biological events which terminate a disease. It is under these circumstances that complement plays its most important role, for it promotes phagocytosis of infectious agents which have combined with antibody and in many instances it mediates their dissolution, or lysis. Not only is complement of great interest in this connection, but also because of its enormous use in diagnostic tests such as the Bordet-Wassermann test for syphilis and numerous tests for viruses which are pathogenic for man and animals. The mechanism of complement uptake or "fixation" in these commonly used and highly crucial tests is still poorly understood, as is also the mechanism of the lysis (hemolysis, in this case) of the red cell-anti-red cell complex used as "indicator" for the presence of complement in the tests.

A conference on complement was accordingly held at Ram's Head Inn, Shelter Island, N. Y., from June 12 to 15, 1950. Those who participated were: A. R. Dochez, H. Eagle, F. Haurowitz, M. Heidelberger, C. W. Hiatt, J. F. Kent, M. H. Loveless, D. A. MacInnes, C. M. MacLeod, F. Maltaner, M. M. Mayer, S. Ochoa, A. G. Osler, A. M. Pappenheimer, Jr., L. Pillemer, E. Ponder, V. R. Potter, C. E. Rice, W. H. Seegers, and A. J. Weil. O. T. Avery, E. E. Ecker, J. W. Fisher and S. B. Hooker were unable to attend.

The topics of the tentative program were: (1) Chemistry of complement and its components. (2) Complement fixation and its relation to antigen-antibody reactions, including neutralization of viruses. (3) Immune hemolysis: kinetics, mechanism, relation to enzymatic processes. (4) Bacteriolysis; enhancement of phagocytosis, immobilization of spirochetes. (5) Complement and the coagulation of the blood.

Actual discussion developed somewhat as follows:

1. Definition of Complement (C′).—Particularly for the benefit of those present who had not actually worked with *C′* an attempt was made to define *C′* but agreement could not be reached. This was mainly because independent, functionally different methods of measurement lead to conflicting results. For example, the diagnostically important "fixation" of *C′*, that is, its uptake by the interaction of antigen and antibody, is measurable as a first approximation by the increase in weight of the antigen-antibody complex.[2] The values so obtained, however, do not run parallel to the decrease observed in the hemolytic power of *C′*,[3] the biological property of *C′* actually measured in the diagnostic test. The importance was emphasized of studies now under way to resolve this discrepancy. It was, however, generally agreed that Muir's old definition, "a fixable constituent of fresh serum"[4] required supplementation by something like "*C′* is a complex of serum constituents which brings about the lysis of red cells, immobilization of spirochetes, the killing of certain Gram-negative bacteria and the opsonization of certain bacteria, all of which functions usually require the cooperation of antibodies to the cells involved." Other activities and manifestations of *C′* were discussed but not included in the definition because of lack of understanding of the underlying phenomena.

2. Chemistry of C′.—Pillemer gave a brief résumé. Of the four components of *C′* (*C′*1, *C′*2, *C′*3, *C′*4) only *C′*1 has thus far been isolated, from guinea pig *C′*, in reasonably pure form as judged by electrophoretic and ultracentrifugal analysis.[5] This work has had far-reaching consequences and the urgency of the isolation of the other components in a state of purity was discussed by the group. Heidelberger told of the separation of *C′*3 as a single activity from pig *C′*, which is usually characterized by a titer of *C′*3 tenfold that of *C′*3 in guinea pig *C′*.

3. Measurement of C′ by Increase in Weight of Specific Precipitates.—The technique and limitations of the method were outlined by Heidelberger. Guinea pig, human, bovine and pig complements all give values of *C′* nitrogen between 0.04 and 0.06 mg. per ml. when volumes of *C′* used are extrapolated to 0. No further light was thrown on the "solubility effect" causing lower values per ml. with larger volumes of *C′*. It was generally agreed that most of the weight added by *C′* is derived from *C′*1.

4. Complement Fixation as Studied by Measurements of Hemolysis.—Recent quantitative studies were described by Osler. While these lead to

curves of C' uptake greatly resembling those of specific precipitation at far higher concentrations of antigen and antibody, the maximum C' uptake indicated by the residual hemolysis may be much greater than can be accounted for on a weight basis if an attempt is made to correlate C' weights and hemolytic titers. Heidelberger presented evidence indicating that because of preferential fixation of $C'1$ and $C'4$ from guinea pig C' measurements made before and after fixation might not be comparable owing to a shift in the limiting components, i.e., those present in lowest titer. These are always $C'2$ and $C'3$ in whole guinea pig C'. It was decided that further work was necessary for general acceptance of this view and that truly quantitative methods for the measurement of component activities were needed since the qualitative methods now available involve large uncertainties. Moreover, fixation of C', as pointed out by Mayer, might be accompanied by an inactivation of C' as well as a loading of C' molecules onto antigen-antibody aggregates.

5. *Mechanism of Hemolysis by Antibody and C'.*—Mayer reviewed some of the older work which suggests that the hemolytic process may be enzymic. Two sets of observations stand out: Mg^{++} is an essential ion[6] and as few as 50 molecules of antibody and 20,000 molecules of C' (a highly uncertain value) are required for lysis of one cell, contrasted with about 10^9 molecules of a surface-active agent, such as saponin. He then described recent kinetic studies[7] which suggest that hemolytic antibody is the enzyme and that C' is an accessory factor used up in the process, perhaps as an energy donor. The experiments have indicated that the antibody-red cell union is reversible, making possible transfer of antibody from cell to cell. This finding is compatible with the view that the antibody is the enzyme, but is not compelling since some simple lysins can undergo transfer as pointed out by Ponder. Mayer has found that with complement in excess, and with antibody the limiting factor, the velocity of lysis is inversely proportional to the total cell concentration (lysed and unlysed). On the other hand, in a system with limited complement and excess antibody, the velocity of lysis is independent of the cell concentration. It is assumed that the lysis of a red cell results from damage to its surface structure and that a cell lyses when damage becomes critical. Therefore, the maximal rate of lysis should be directly proportional to the rate of production of damage (or strain) at the surface of the cell. (This is an approximation; strict analysis involves probability theory). From the rate of damage production in the whole system $\left(\frac{dS}{dt}\right)$ one obtains the rate of damage production per cell by dividing by the total number of cells; the maximal rate of lysis would then be: $-\frac{dE}{dt} = K\frac{(dS)}{(dt)} \cdot \frac{1}{E_0} \cdot E$, where E_0 = total cells and E = unlysed cells. Accordingly, if antibody is considered to be the en-

zyme, and conditions are such that antibody is limited (i.e., substrate is in excess), the enzyme is working at top speed; therefore, variations of substrate concentration (i.e., cells) do not affect the rate at which damage is produced, but do affect the maximal rate of lysis in inverse proportion. On the other hand, when antibody as the enzyme is in excess, the rate of damage production will be proportional to E_0, and therefore the maximal rate of lysis will be independent of the cell concentration. Since these kinetic deductions are verified by experimental results obtained with hemolytic antisera exhibiting relatively rapid transfer of antibody from cell to cell, they constitute an argument that the antibody acts like an enzyme, and complement is a substance or factor which is used up.

After returning to the University of Wisconsin, Potter discussed the kinetics of hemolysis with Alberty, who had worked out a mathematical treatment of a somewhat analogous but simpler situation involving the reactivity of ϵ-amino groups on proteins. The discussion resulted in the formulation of a theory of immune hemolysis by Alberty and Baldwin. This will probably be published after the conclusion of correspondence with Mayer.

On the last day of the conference Ponder agreed, with the help of R. T. Cox, to attempt the preparation of a clear and comprehensive statement about the various schemes which have been proposed for the kinetics of hemolysis. There has also been considerable correspondence between Ponder and Alberty, as Ponder conceives of Alberty's theory as a special case of the more general theory which he and Cox expect soon to publish.

In connection with the enzyme function Pillemer pointed out that earlier studies with Ecker indicated that $C'3$ is not used up in hemolysis[8] and that he believes for this reason that $C'3$ is the enzymic constituent of this system. Haurowitz considered it possible that an antibody-complement complex might have enzymic functions.

6. Relation of C' to Phenomena of Blood Coagulation.—Seegers reviewed the present status of the mechanism of blood coagulation and expressed the belief, to which Pillemer agreed, that prothrombin is not identical with $C'1$. However, accelerator globulin and C' components may be related and this deserves further exploration.

Maltaner reported on the effects of cephalin on blood coagulation and C' activity,[9] but without any clear evidence that these processes are directly related.

General discussion followed of the role of Ca^{++} and Mg^{++} in blood coagulation and C' activity. The two systems differ in that Ca^{++} appears to be the crucial constituent of the blood clotting system, whereas Mg^{++} is the more important ion for the lytic activity of C'.

7. Conglutination.—Rice reported some of her recent work. The conglutination reaction is often useful in diagnostic tests involving bird and animal

sera. It involves the agglutination of sheep erythrocytes by heated bovine serum and fresh horse serum and can be used as an indicator system in the complement fixation test since it shows analogies to hemolysis. The heated bovine serum contributes a substance called conglutinin while the fresh horse serum is thought to contribute C' or some of its components. If a specific antigen-antibody reaction occurs it fixes the horse C' and thus prevents conglutination. Several members of the group challenged the view that the action of the fresh horse serum is due to complement, so that further study of this complex subject was indicated.

The conference adjourned with all members much stimulated by the contacts made and by the uninhibited discussions. These reflected approaches to the subject from many entirely different points of view and usually centered less about what is known of C' than about the defects and inconsistencies in the information presently available. Ways and means were discussed in detail for obtaining some of the most urgently needed data. As Haurowitz remarked, all of the participants will in future consider C' from more critical and more objective viewpoints.

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